# Probabilisic Methods in Combinatorics
# Exercise Sheet 8

**Question 1.** As in the notes, show that if there exists $p, x, y \in (0, 1)$ such that

$$p^3 \leq x(1-x)^{3n}(1-y)^{\binom{n}{k}} \text{ and } (1-p)^{\binom{k}{2}} \leq y(1-x)^{\frac{nk^2}{2}}(1-y)^{\binom{n}{k}}$$

then $R(3, k) > n$.

(* What about for general $l, k$?)

**Question 2.** Let $H = (V, E)$ be a hypergraph in which every edge has at least $k$ elements, and suppose that each each intersects at most $d$ other edges.

Show that, if $e(d + 1) \leq 2^{k-1}$, then we can 2-colour $V$ such that no edge is monochromatic.

A hypergraph is $k$-uniform if every edge has size $k$ and $k$-regular if every $v$ lies in $k$ edges. Deduce that the same holds true for any $k$-uniform, $k$-regular hypergraph, as long as $k \geq 9$.

**Question 3.** The $k$-SAT problem can be roughly defined as follows. There is some collection of variables $x_i$ which can take the values 0 or 1. You are given some set of statements of the form

$$S_i = x_{i_1} \vee \neg x_{i_2} \vee \ldots \vee x_{i_k}$$

all involving $k$ variables which can either be $x_i$ or $\neg x_i$ for some $i$. The task is to find some assignment for the variables such that all of the statements are true (that is, equal to 1). The $k$-SAT problem is deciding whether or not this is possible for a given set of statements.

Suppose we have a set of statements such that each $x_i$ (or it's negation) appears in at most $2^{k-2}/k$ statements, show we can find such an assignment.

**Question 4.** Let $G = (V, E)$ be a bipartite graph with $n$ vertices, and suppose that for each vertex we are given some list $S(v)$ of colours such that $|S(v)| \geq \log_2(n)$ for each $v$.

Show using basic probabilistic methods that there is a proper colouring of $G$ such that the colour of each vertex $v$ is in $S(v)$.

Given any graph $G = (V, E)$, and for each vertex a list $S(v)$ of colours such that $|S(v)| \geq 10d$ for each $v$. Suppose further that for each $v \in V$ and $c \in S(v)$ there are at most $d$ neighbours, $u$, of $v$ with $c \in S(u)$.

Show that again we can find a proper colouring of $G$ such that the colour of each vertex $v$ is in $S(v)$.