Probabilisic Methods in Combinatorics
Exercise Sheet 7

Question 1. Show that almost surely the maximum (and minimum) degree of $G(n,p)$ for fixed p is

$$pn + O(\sqrt{n \log(n)})$$

Question 2. Given an edge e in the edge set of $G(n,p)$ let X_e be the number of triangles in $G(n,p)$ containing e.

For any fixed $\epsilon > 0$ show that, if $p = \omega\left(\sqrt{\frac{\log(n)}{n}}\right)$ then with high probability $X_e = (1 \pm \epsilon)p^2 n$.

By keeping track of the probability in the previous part show that, with high probability, every edge is in approximately the same number of triangles.

Question 3. Suppose $\epsilon > 0$ and let $d \geq c/\epsilon^2$ for some large constant c. Show that with high probability in $G(n, d/n)$ for every partition $V = A \cup B$, with $|A| = |B|$ the number of edges between A and B is between $(1-\epsilon)\frac{dn}{4}$ and $(1+\epsilon)\frac{dn}{4}$.

Deduce that for any $\delta > 0$ there is some d' such that there exists a graph with average degree d' such that every equipartition $V = A \cup B$ has between $(1-\delta)\frac{d'n}{4}$ and $(1+\delta)\frac{d'n}{4}$ edges between A and B.

Question 4. Suppose we place n points uniformly at random in $[0,1]^2$. If we consider $[0,1]^2$ as being composed of $\frac{n}{\log(n)}$ equally sized disjoint squares (ignoring the rounding issues), show that with high probability there is at least 1 point in each square.

Using the Chernoff bound show that for each point, with high probability, the number of points within distance $10\sqrt{\frac{\log(n)}{n}}$ is at most $c\log(n)$ for an appropriate constant c.

Suppose we form a graph G on these n points by joining each point to the nearest k points to it. Show that, if $k \geq c\log(n)$ then with high probability G is connected.