Probabilisic Methods in Combinatorics
Exercise Sheet 6

Question 1. Show that $r(n) = \frac{1}{n}$ is a threshold function for the property of $G(n,p)$ containing a cycle (of any length).

Question 2 (Clique Number of $G(n,p)$)**.** Let

$$f(k) = \binom{n}{k} 2^{-\binom{k}{2}}.$$

What is significant about this number? Show that roughly $f(k)$ is around $n^k 2^{-\frac{k^2}{2}}$, and that when k is smaller than $(2-\epsilon)\log_2(n)$, $f(k) \to \infty$, and similarly if k is bigger than$(2+\epsilon)\log_2(n)$, $f(k) \to 0$.

(No need to be too rigorous, just to get a feel of how this quantity behaves).

Question 3. For each k-set $S \subset [n]$ let X_S be the indicator random variable of the event that S is a clique in $G(n,1/2)$ and let $X = \sum X_S$ be the number of cliques. Show that

$$\mathrm{Var}(X) \leq f(k) \left(\sum_{i=2}^{k} \binom{k}{i} \binom{n-k}{k-i} 2^{\binom{i}{2} - \binom{k}{2}} \right).$$

Question 4. So to apply Chebyshev's to the above quantity we want to estimate the terms

$$g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i} 2^{\binom{i}{2}-\binom{k}{2}}}{f(k)} = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}.$$

Suppose k is sufficiently 'small' (at least such that $f(k) \geq n^3$ as $n \to \infty$). We will assume further that all the other $g(i)$ are asymptotically negligible compared to $g(2)$ and $g(k)$. Show that both $g(2)$ and $g(k)$ are $o(1)$.

(Hint: Consider the ratio $g(2)/g(k)$)

Deduce by Chebyshev's inequality that with high probability $\omega(G) \geq k$.

Question 5. Let H be a graph and $n > |V(H)|$ be an integer. Suppose that there exists some graph G on n vertices with t edges such that $kt > n^2 \log(n)$ and G does not contain a copy of H.

Show that we can k-colour K_n such that there is no monochromatic copy of H.