Probabilisic Methods in Combinatorics
Exercise Sheet 5

Question 1. Let $t > 0$. Find a random variable X with $\mathrm{Var}(X) < \infty$ such that equality holds in Chebyshev's inequality, that is such that

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) = \frac{\mathrm{Var}(X)}{t^2}.$$

Question 2. Let X be a random variable with $\mathbb{E}(X) = 0$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Show that for all $\lambda > 0$

$$\mathbb{P}(X \geq \lambda) \leq \frac{\sigma^2}{\lambda^2 + \sigma^2}.$$

Question 3. Consider the model of a random bipartite subgraph $G(n, n, p)$ on two equal size vertex classes of size n, where each edge is included in the graph independently with probability p.

Show that the function $r(n) = \frac{1}{n}$ is a threshold function for the event that the graph contains a 4-cycle (that is, a complete bipartite graph $K_{2,2}$).

Question 4. We say a set $\{x_1, x_2, \ldots, x_k\}$ of positive integers has distinct sums if for all subsets $I \subset [k]$ the quantities

$$\sum_{i \in I} x_i$$

are distinct. Let $f(n)$ be the maximum size of a set $\{x_1, x_2, \ldots, x_k\} \subset [n]$ with distinct sums.

Show combinatorially that $f(n) \leq \log_2(n) + \log_2(\log_2(n)) + O(1)$ and (by exhibiting such a subset of $[n]$) that $f(n) \geq \lceil \log_2(n) \rceil$.

By picking a random subset I of $[k]$ by including each index with probability $1/2$ and considering the sum

$$\sum_{i \in I} x_i$$

show that $f(n) \leq \log_2(n) + 1/2 \log_2(\log_2(n)) + O(1)$

Question 5. Given any graph H a 1-*division* of H is the graph obtained by replacing each edge with a path of length 2. Note that a 1-division is a bipartite graph such that all the vertices in once class have degree bounded by 2.

Let G be a graph with ϵn^2 edges for some $\epsilon > 0$. Show that G contains a 1-division of K_p for some $p = \Omega(n^{\frac{1}{2}})$.

(Hint: We can in fact take $p = \epsilon^{\frac{3}{2}} n^{\frac{1}{2}}$, which may make the some of the estimates clearer to follow.)