Probabilisic Methods in Combinatorics
Exercise Sheet 4

Question 1. Let G be a graph with n vertices and m edges. Show that the exists a bipartite subgraph of G with at least $m/2$ edges.

(* Improve this to show that when G has $2n$ vertices we can find a bipartite subgraph with at least $m\frac{n}{2n-1}$ edges and when G has $2n+1$ vertices $m\frac{n+1}{2n+1}$)

Question 2. Given a random graph $G(n,p)$ let X be the random variable which counts the number of edges in G and Y be that which counts the number of triangles. By choosing a suitable p, use the alteration method to show that there exists a graph on n vertices with $\frac{1}{6}n^{3/2}$ edges which contains no triangle.

Is this best possible?

Question 3. Let H be a 3-uniform hypergraph with n vertices and $m \geq n/4$ edges. Show that H contains an independent set of size at least

$$\frac{3n^{\frac{3}{2}}}{8\sqrt{m}}.$$

(3/8 is not optimal, but hopefully should make the calculations easier, feel free to prove a better bound)

Question 4 (The Szemerédi Trotter Theorem)**.** Let P be a collection of n distinct points in $\mathbb{R}^2$ and let L be a collection of m distinct lines. Let

$$I = \{(p,l) \,:\, p \in P, l \in L \text{ and } p \in l\}$$

be the collection of point-line incidences. Show that

$$|I| = O(m^{\frac{2}{3}}n^{\frac{2}{3}} + m + n)$$

(Hint: Use Theorem 3.8)

Can you find an example to show that this is sharp?

Question 5. Let A be a set of n real numbers. Consider the set of sums $A+A = \{a+b \,:\, a,b \in A\}$ and products $A.A = \{ab \,:\, a,b \in A\}$.

Show that $2n-1 \leq |A+A|, |A.A| \leq \binom{n}{2} + n$. Find examples of sets achieving the lower bound for each.

Show that, for any A, at least one of $|A+A|$ or $|A.A|$ is greater than $\epsilon n^{5/4}$ for an appropriately small ϵ.

(Hint: Consider the lines $ax - ab$ for $a,b \in A$)