Probabilisic Methods in Combinatorics
Exercise Sheet 3

Question 1. Show that for a real random variable X on a finite probability space which takes values in V_X,

$$\mathbb{E}(X) := \sum_{\omega\in\Omega} X(\omega)\mathbb{P}(\omega) = \sum_{a\in V_X} a\mathbb{P}(X = a).$$

Question 2. Let $\sigma \in S_n$ be a permutation chosen uniformly at random. $i \in [n]$ is a *fixed point* of σ if $\sigma(i) = i$. What is the expected number of fixed points of σ?

Give a lower bound for the probability of the event

$$\mathbb{P}(\sigma \text{ has at most } k \text{ fixed points}).$$

Question 3. Let X be a random variable taking integer values on $\mathcal{G}(n, p)$. Show that if $\mathbb{E}(X) \to 0$ then

$$\mathbb{P}(X = 0) \to 1.$$

(that is, with high probability X is 0).

Show that if $p = o(n^{-2/3})$ then with high probability $G(n, p)$ contains no cliques of size 4.

Suppose that $\mathbb{E}(X) \to \infty$, can we deduce that with high probability X is 'large'? (For example, even that $\mathbb{P}(X = 0) \to 0$?)

Question 4 (LYM inequality and Sperner's Theorem)**.** Let $\mathcal{A} \subset 2^{[n]}$ be an anti-chain, that is, a set of subsets $A_i \subset [n]$ such that there is no pair satisfying $A_i \subset A_j$. Let $a_k = |\mathcal{A} \cap [n]^{(k)}|$ be the number of k-sets in $\mathcal{A}$. Show that

$$\sum_{k=1}^{n} \frac{a_k}{\binom{n}{k}} \leq 1.$$

(Hint: Pick a random ordering $x_1 < x_2 < \ldots < x_n$ and consider the sets $C_i = \{x_1, x_2, \ldots, x_i\}$.)

Using this show that

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Is this best possible?

Question 5. A set A contained in a group is said to be sum-free if there is no triple $a_1, a_2, a_3 \in A$ such that $a_1 + a_2 = a_3$. Show that, in $\mathbb{Z}_p$, the interval $[p/3, 2p/3]$ is sum-free.

Show that every set B of n non-zero integers contains a sum-free subset A of size $|A| > \frac{1}{3}n$.

(Hint: Consider a random translate $x.B = \{xb : b \in B\}$ inside $\mathbb{Z}_p$ for a large prime p.)

*Show that further we can find disjoint sum-free subsets A_1 and A_2 such that $|A_1| + |A_2| > \frac{2}{3}n$.