Probabilisic Methods in Combinatorics
Exercise Sheet 2

Question 1. Show that
$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Question 2. Show that if we are more careful in our estimations the proof of Theorem 2.2 will show that, for the diagonal Ramsey numbers, there is some constant $c > 0$ such that.
$$R(k,k) \geq ck2^{\frac{k}{2}}.$$

(What's the best value of c you can get? A slight improvement can be gotten by using $\binom{n}{k} \leq n^k/k!$ and Stirling's approximaton that $k! \sim \sqrt{2\pi k}(k/e)^k$.)

Question 3. Show that for general k, l, if there exists p such that
$$\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1,$$
then $R(k,l) > n$ and using this show
$$R(4,l) \geq \Omega\left(\frac{l^{\frac{3}{2}}}{(\log(l))^{\frac{3}{2}}}\right).$$

(* What about general k and l, $l >> k$?)

Question 4. Let $n \geq 4$ and let H be an n-uniform hypergraph with at most $4^{n-1}/3^n$ edges. Show that there is a colouring of the vertices of H using 4 colours such that each colour appears in every edge.

Question 5. Suppose we have two families of sets $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$, as in the notes, which satisfy the following properties

- $|A_i| = k$, $|B_i| = l$ for all $1 \leq i \leq n$;
- $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq n$;
- Either $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$ for all $1 \leq i < j \leq n$.

show that
$$n \leq \frac{(k+l)^{k+l}}{k^k l^l}$$

(Hint: Consider $\mathcal{A}$ and $\mathcal{B}$ as being contained in $[m]$ for m very large. For any $p < 1$ consider the event that a randomly picked subset of size pm contains A_i and avoids B_i. Take the limit as $m \to \infty$.)