Probabilisic Methods in Combinatorics
Exercise Sheet 12

Question 1. Given a fixed $\alpha \leq 1/2$, by considering the set $\mathcal{C}$ of all subsets of $[n]$ of size at most αn show that

$$\sum_{i \leq \alpha n} \binom{n}{i} \leq 2^{H(\alpha)n},$$

with $H(\alpha)$ defined as in the notes.

(Hint: Pick a random X from $\mathcal{C}$, associate it with it's characteristic vector and then estimate it's entropy)

Question 2 (Combinatorial Shearer's Lemma)**.** Let $\mathcal{F}$ be a family of (not necessarily distinct) subsets of $[n]$ such that each $i \in [n]$ is in at least t sets in $\mathcal{F}$. Let $\mathcal{A}$ be another family of subsets of $[n]$ and let us define, for every $F \subset [n]$,

$$\text{trace}_F(\mathcal{A}) = \{A \cap F \, : \, A \in \mathcal{A}\}.$$

Using Shearer's lemma show that

$$|\mathcal{A}| \leq \prod_{F \in \mathcal{F}} |\text{trace}_F(\mathcal{A})|^{\frac{1}{t}}.$$

Question 3. Suppose we have an arbitrary graph G on an even number of vertices, show the the union of the edge sets of a pair of matchings in G gives a subgraph consisting of a set of isolated edges and cycles. For each such subgraph calculate the number of pairs of matchings of which it is the union.

By picking an arbitrary orientation on each cycle, use such a subgraph to define a matching in $G \times K_2$. Hence show that there is an injection from $\Phi(G) \times \Phi(G)$ to $\Phi(G \times K_2)$.

Therefore show that for any graph G with an even number of vertices we have that

$$|\Phi(G)| \leq \prod_{v \in V} (d(v)!)^{\frac{1}{2d(v)}}.$$

Question 4. Let $\mathcal{G}$ be a set of graphs on $[n]$ such that for every pair of graph $G_i, G_j \in \mathcal{G}$ there is an edge in their intersection $G_i \cap G_j$. How large can $\mathcal{G}$ be?

Suppose instead that we insist that there is a triangle in each intersection. Give an explicit example of a family of size $2^{\binom{n}{2}-3}$ with this property.

Let us presume for ease of presentation that n is even. We consider each G_i as a subset of the set $U = \binom{n}{2}$, and for each equipartition $[n] = A \cup B$ such that $|A| = |B|$, let $U(A,B)$ be the set of edges which lie entirely in A or B.

Show that the trace of $\mathcal{G}$ on any $U(A,B)$ is an intersecting family. Hence by considering the trace of $\mathcal{G}$ over the family $\mathcal{F} = \{U(A,B) \, : \, (A,B) \text{ an equipartition}\}$ show that

$$|\mathcal{G}| \leq 2^{\binom{n}{2}-2}.$$