Probabilisic Methods in Combinatorics
Exercise Sheet 11

Question 1. Consider an arbitrary graph G, and form a subgraph $H \subseteq G$ by placing each edge in $E(H)$ independently with probability p. Let X be the number of $v \in V(G)$ which meet an edge in H. Show that X is tightly concentrated about its mean.

(Hint: First show that it is concentrated about its median)

Question 2. Suppose we have some weight $x_1, \ldots, x_n \in [0,1]$ and we want allocate them to bins that can hold things of total weight 1. Let $B(x_1, \ldots, x_n)$ be the smallest number of bins. Show that $B(x_1, \ldots, x_n) \leq 2\sum_i x_i + 1$.

For any $a \in \mathbb{N}$ let $A(a) = \{y : B(y) \leq a\}$. Show that for any $x \in [0,1]^n$ and $a \in \mathbb{N}$

$$B(x) \leq a + 2||x||_2 d(A(a), x) + 1,$$

where d is the generalised hamming distance and $||x||_2$ is the euclidean norm of x.

Suppose we pick the $x = (x_1, \ldots, x_n)$ independently and uniformly at random from $[0,1]^n$. Show that with high probability $||x||_2^2$ is concentrated about its mean, and hence that with high probability $||x||_2$ is concentrated about $\sqrt{n}2$.

Deduce that with high probability

$$B(x) \leq m(B) + 2\sqrt{\log n}\sqrt{n}.$$

Question 3. Suppose we are working in the probability space $\mathcal{G}(n,p)$ we consider the random variable P which counts the number of paths with 2 edges in $G(n,p)$. Calculate the expected value of P and show that if $p = n^{-\frac{3}{2}+\beta}$ for $\beta > 0$ that $\mathbb{E}(P) \to \infty$.

Use the Azuma-Hoeffding inequality to show that, if $p = n^{-\frac{1}{2}+\gamma}$, then P is tightly concentrated about its mean.

Furthermore show that, if $p \leq n^{-\frac{1}{2}+\gamma}$ then

- $\mathbb{P}(P \geq 2n^{2+2\gamma})$ is exponentially small;
- $\mathbb{P}(\text{Any edge is in } \geq 4n^{\frac{1}{2}+\gamma} \text{ paths of length 2})$ is exponentially small.

Using this, show that when $p = n^{-\frac{3}{4}+\gamma'}$ then P is still tightly concentrated about its mean.

(* Continue the bootstrapping process to get get concentration for $p = n^{-\frac{3}{2}+\beta}$ for arbitrary $\beta > 0$.)