Probabilisic Methods in Combinatorics
Exercise Sheet 10

Question 1. Show the following generalization of Lemma 10.4: Suppose (Z_i) is a sequence of mutually independent random variables on a probability space and A is a random variable on the same space. Suppose further that changing the value of one Z_i can change the value of A by at most c.

Define X_i as in Lemma 10.1, and show that it satisfies the boundedness condition

$$|X_{i+1} - X_i| \leq c$$

Question 2. Suppose we flip n biased coins, each which land heads with probability p and tails with probability $1 - p$, independently of the others. Explain how we can express the above situation as a probability space on $\{0, 1\}^n$ and, given an arbitrary random variable X on this probability space, define the associated co-ordinate exposure martingale.

If we let A be the number of heads flipped, show that A is a 1-Lipschitz function (as in Section 11) and, assuming the previous question, use the Azuma-Hoeffding inequality to bound, for any $t \geq 0$, the probability

$$\mathbb{P}(|A - \mathbb{E}(A)| \geq t).$$

Compare this to the Chernoff bounds.

Question 3. Suppose we throw n balls into m bins independently and uniformly at random. Let E be the number of empty bins at the end, what is $\mathbb{E}(E)$?

Let E_j be the indicator function for the event that the jth bin is empty, so that $E = \sum_{j=1}^{m} E_j$. Are the E_j independent?

Let Z_i be the bin into which the ith ball falls into and consider the martingale given by L with respect to the sequence $(Z_i)_1^n$ (as in Lemma 10.1). Show that

$$\mathbb{P}(|E - \mathbb{E}(E)| \geq \lambda\sqrt{n}) \leq 2e^{-\frac{\lambda^2}{2}}.$$

and so conclude that E is tightly concentrated about it's mean if $m = n$.

Question 4. Let Q_n be the n-dimensional hypercube . Given a set $A \subset \{0, 1\}^n = V(Q_n)$ and $s \in \mathbb{N}$ let $B(A, s)$ be the set of vertices whose Hamming distance is less than s to A, that is, for every $x \in B(A, s)$ there exists a $y \in A$ such that x and y differ on less than s coordinates.

Let $\epsilon, \lambda > 0$ be such that $e^{-\frac{\lambda^2}{2}} = \epsilon$ and suppose $|A| \geq \epsilon 2^n$. Show that

$$|(B(A, 2\lambda\sqrt{n})| \geq (1 - \epsilon)2^n.$$