Discrete Entropy
Exercise Sheet 8

Question 1 (Combinatorial Shearer's Lemma)**.** Let $\mathcal{F}$ be a family of (not necessarily distinct) subsets of $[n]$ such that each $i \in [n]$ is in at least t sets in $\mathcal{F}$. Let $\mathcal{A}$ be another family of subsets of $[n]$ and let us define, for every $F \subset [n]$,

$$\text{trace}_F(\mathcal{A}) = \{A \cap F \,:\, A \in \mathcal{A}\}.$$

Using Shearer's lemma show that

$$|\mathcal{A}| \leq \prod_{F \in \mathcal{F}} |\text{trace}_F(\mathcal{A})|^{\frac{1}{t}}.$$

Question 2. Let G be a graph with $e(G) = m$ edges. Let $X = (X_1, X_2, X_3)$ be a random variable which is uniform on the subset $T \subseteq V(G)^3$ where

$$T = \{(a, b, c) \colon a, b, c \text{ distinct and } G[a, b, c] \text{ is a triangle}\}.$$

By considering the collection $\{(1, 2), (1, 3), (2, 3)\}$, use Shearer's lemma to give a bound on the number of triangles in G.

Question 3. Let $\mathcal{G}$ be a set of graphs on $[n]$ such that for every pair of graphs $G_i, G_j \in \mathcal{G}$ there is an edge in their intersection $G_i \cap G_j$. How large can $\mathcal{G}$ be?

Suppose instead that we insist that there is a triangle in each intersection. Give an explicit example of a family of size $2^{\binom{n}{2}-3}$ with this property.

Let us presume for ease of presentation that n is even. We consider each G_i as a subset of the set $U = \binom{n}{2}$, and for each equipartition $[n] = A \cup B$ such that $|A| = |B|$, let $U(A, B)$ be the set of edges which lie entirely in A or B.

Show that the trace of $\mathcal{G}$ on any $U(A, B)$ is an intersecting family. Hence by considering the trace of $\mathcal{G}$ over the family $\mathcal{F} = \{U(A, B) \,:\, (A, B) \text{ an equipartition}\}$ show that

$$|\mathcal{G}| \leq 2^{\binom{n}{2}-2}.$$

Question 4. Suppose $\mathcal{G}$ is set of graphs on $[n]$ such that for every pair of graphs $G_i, G_j \in \mathcal{G}$ the intersection $G_i \cap G_j$ contains no isolated vertices. Prove an upper bound for $|\mathcal{G}|$ and show that it is tight.

Question 5. Let $\mathcal{A}$ be a multiset of non-empty subsets of $[n]$. Given a pair A_i, A_j of non-nested (that is, neither $A_i \subseteq A_j$ or $A_j \subseteq A_i$) sets in $\mathcal{A}$ let $\mathcal{A}'$ be the set obtained by replacing A_i and A_j with $A_i \cap A_j$ and $A_i \cup A_j$ (keeping only the latter if the former is empty). We call $\mathcal{A}'$ and elementary compression of $\mathcal{A}$ and we define a partial order $<$ on multisets by letting $\mathcal{A} < \mathcal{B}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ via a sequence of elementary compressions.

Show that $<$ is a partial order and show that there is a unique minimal multiset $\mathcal{A}^{\#}$ dominated by $\mathcal{A}$.

Let $X = (X_1, \ldots, X_n)$ be a random variable and let $\mathcal{A} < \mathcal{B}$. Show that

$$\sum_{A \in \mathcal{A}} \mathbb{H}(X_A) \leq \sum_{B \in \mathcal{B}} \mathbb{H}(X_B).$$

Show that Shearer's inequality follows from the above.