Discrete Entropy
Exercise Sheet 7

Let T be a tree rooted at $r \in V(T)$ and consider the following method of generating an element of $\hom(T, G)$: Fix an order $r = v_0, v_1, \ldots, v_t$ of the vertices of T such that every vertex appears before all it's children. Choose $f(r)$ according to the distribution D from the lectures, i.e $\mathbb{P}(f(r) = v) = \frac{d(v)}{2e(G)}$ for each $v \in V(G)$. Then for each $i > 1$ we sequentially choose $f(v_i)$ uniformly from the neighbours of $f(w)$, where w is it's parent. This defines a random variable X_T on $\hom(T, G)$.

Question 1. Show that X_T doesn't depend on the choice of r or the ordering $v_0, v_1, \ldots, v_t$.

(Hint: For a fixed $f \in \hom(T, G)$ calculated $\mathbb{P}(X_T = f)$.)

Given a sub-tree $S \subseteq T$ let Y be the marginal distribution of S in X_T. Show that $Y \sim X_S$.

Question 2. Show that X_T is a witness variable for T.

(It may help to consider X as $(X_0, \ldots, X_t)$ where $X_i = X(v_i)$ is the image of v_i under X)

Question 3. Given two directed graphs H and G one can define the homomorphism density of H in G in the same way as before. Consider the two graphs $\overrightarrow{C}_3$, a directed triangle and V, a graph on vertex set $\{x, y, z\}$ with edge set $\{(x, y), (x, z)\}$.

Let X be a uniform random variable on $\hom(\overrightarrow{C}_3, G)$ with marginal distributions (X_1, X_2, X_3) for the vertices. Show that

$$\log(\hom(\overrightarrow{C}_3, G)) \leq \mathbb{H}(X_1) + 2\mathbb{H}(X_2|X_1).$$

Construct a random variable Y on $\hom(V, G)$ such that $\mathbb{H}(Y) = \mathbb{H}(X_1) + 2\mathbb{H}(X_2|X_1)$. Deduce that $\hom(V, G) \leq \hom(\overrightarrow{C}_3, G)$ for every graph G.

Question 4. Suppose that if F is a forest and Y is a random variable taking values in $\hom(F, G)$ such that the marginal of Y on every edge of F is E and the marginal of Y on every vertex is D (see the lecture notes), then

$$\mathbb{H}(Y) \leq e(F)\mathbb{H}(E) + (v(F) - 2e(F))\mathbb{H}(D).$$

Question 5. Let a *strong witness variable* for a graph H be one that satisfies 1. and 2. from the notes and also the following inequality:

$$\mathbb{H}(X) \geq e(H)\mathbb{H}(E) + (v(H) - 2e(H))\mathbb{H}(D).$$

Show that this implies X is witness variable for H (If H has no isolated vertices).

Let H_1 and H_2 be graphs with strong witness variables X_1 and X_2 and let $S_1 \subseteq V(H_1)$ and $S_2 \subseteq V(H_2)$. Suppose there is a bijection $g : S_1 \to S_2$ such that the marginal distribution of S_1 in X_1 is the same as the marginal distribution of $g(S_1)$ in X_2, which we will denote by X_S. Furthmore suppose that S_1 and S_2 both span forests in H_1 and H_2 respectively, and that g is an isomorphism on these forests.

Let $H = H_1 \oplus_g H_2$ and let X be the conditionally independent coupling of X_1 and X_2 over X_S. Then X is a strong witness variable for H.

(**Show that the hypercube Q_d is Sidorenko for each d.)