Discrete Entropy
Exercise Sheet 6

Question 1. Let X be distributed as $\text{Bin}(n, p)$ and Y as $\text{Bin}(m, q)$ where $n \leq m$ and $p \leq q$. By giving a coupling of X and Y show that for any $a \in \mathbb{R}^+$

$$\mathbb{P}(X \geq a) \leq \mathbb{P}(Y \geq a).$$

Question 2. Suppose P and Q are distributions on the same set $\mathcal{X}$ and we write $p(x)$ and $q(x)$ for $\mathbb{P}(P = x)$ and $\mathbb{P}(Q = x)$ respectively. The *Kullback-Leibler divergence* of P and Q is given by

$$D_{KL}(P||Q) = \sum_{x \in \mathcal{X}} p(x) \log \left(\frac{p(x)}{q(x)} \right).$$

- Show that $D_{KL}(P||Q) \geq 0$.
- If X takes values on $\mathcal{X}$ and U is uniformly distributed on $\mathcal{X}$ show that $\mathbb{H}(X) = \log |\mathcal{X}| - D_{KL}(X||U)$.
- Given X and Y, let $(\hat{X}, \hat{Y})$ be independent with $\hat{X} \sim X$ and $\hat{Y} \sim Y$. Show that

$$I(X;Y) = D_{KL}(X, Y || \hat{X}, \hat{Y}).$$

Question 3. Suppose P and Q are distributions on the same set $\mathcal{X}$. Let $||P - Q||_1 = \sum_{x \in \mathcal{X}} |p(x) - q(x)|$. Suppose that P and Q are distributed as Bernoulli random variables with success probability p and q respectively. Show that

$$D_{KL}(P||Q) \geq \frac{1}{2 \log_e 2} ||P - Q||_1^2.$$

Question 4 (Pinsker's inequality)**.** Suppose P and Q are distributions on the same set $\mathcal{X}$. Let $A = \{x \colon p(x) \geq q(x)\}$ and let P_A and Q_A be distributed as Bernoulli random variables with success probability $p(A)$ and $q(A)$ respectively. Show that $||P - Q||_1 = ||P_A - Q_A||_1$.

(*Furthermore, show that $D_{KL}(P||Q) \geq D_{KL}(P_A||Q_A)$. Hint: Develop a chain rule for KL-divergence)

(Assuming *) Deduce that the bound in Question 2 holds for arbitrary P and Q. Show that if X is a random variable taking values on $\mathcal{X}$ with $\log |\mathcal{X}| - \mathbb{H}(X) \leq \epsilon$ and U is uniform on $\mathcal{X}$, then $||X - U||_1 \leq f(\epsilon)$ for some function f which tends to 0 with ϵ.

Question 5. Suppose $G_1, \ldots, G_t$ are bipartite graphs on vertex set $[n]$ with partition classes (A_i, B_i) such that $\bigcup_i G_i = K_n$. Show that $t \geq \log(n)$.

(Hint: Let X be a uniformly chosen vertex and consider χ_i, the indicator random variable of the event that $X \in B_i$.)

* Let $\text{size}(G_i)$ be the number of non-isolated vertices in G_i. Show that in fact the stronger bound holds

$$\frac{1}{n} \sum_{i=1}^{t} \text{size}(G_i) \geq \log n$$