Discrete Entropy
Exercise Sheet 5

Question 1. Let $0 \leq p \leq 1/2$. Show that

$$\sum_{i \leq pn} \binom{n}{i} \leq 2^{h(p)n}.$$

(*) Show further, that if $\mathcal{C} \subseteq 2^{[n]}$ and $0 \leq p \leq 1$ is the average of $|C|/n$ over $C \in \mathcal{C}$ then $|\mathcal{C}| \leq 2^{h(p)n}$.

Question 2. Suppose we have n coins, some of which weigh a grams, and some of which are counterfeit and weight $b < a$ grams. We have a set of scales on which we can weigh any subset of the coins.

Suppose we have to choose a sequence of subsets to weigh beforehand, so that we cannot use the partial information to inform our future choices, what is the smallest number needed to find all the counterfeit coins?

(Hint: Let $\{A_1, \ldots, A_m\}$ be the subsets chosen, X be a random subset of $[n]$ and $X_i = |A_i \cap X|$ for each i)

Question 3. Given a graph $G = (V, E)$ with $|V| = 2n$ consider the permanent of it's adjacency matrix A. Show that every permutation σ of $[2n]$ with non-zero contribution to $\text{perm}(A)$ corresponds to a cover of the vertices of G by cycles and isolated edges, which we call a cycle cover.

Conversely, given a pair of perfect matchings M_1 and M_2 of G show that $M_1 \cup M_2$ is a subgraph of G covering the vertices with even length cycles and isolated edges, which we call an even cycle cover.

Using the above show that $|\Phi(G) \times \Phi(G)| \leq \text{perm}(A)$ and show that

$$\phi(G) \leq \prod_{v \in V} (d(v)!)^{\frac{1}{2d(v)}}.$$

Question 4. Consider the 'obvious' notion of entropy extended to continuous random variables X, with probability density function f, taking values in $\mathbb{R}^n$ where

$$\mathbb{H}(X) = -\int f(x) \log f(x) dx$$

where the integral is with respect to the Lebesgue measure.

- Show that $\mathbb{H}(X)$ can take negative values;
- Given $a \in \mathbb{R}^+$ compare $\mathbb{H}(X)$ and $\mathbb{H}(aX)$;
- Suppose X takes values in $[0, 1]$, show that the entropy is maximised with X uniform;
- (* Does conditioning always reduce the entropy?).

(You may assume pretty much whatever you want about Jensen's inequality, and not worry too much about exchanging order of integration)