Discrete Entropy
Exercise Sheet 4

Question 1. Calculate the capacity of the binary symmetric channel and binary erasure channel.

Question 2. Show that the Huffman coding is optimal.

Question 3. A sequence of random variables $X_1, X_2, \ldots, X_n$ forms a Markov chain, which we write as $X_1 \to X_2 \to \ldots \to X_n$, if $p(x_{i+1}|x_1, x_2, \ldots, x_i) = p(x_{i+1}|x_i)$ for every i and every $(x_1, \ldots, x_i)$ in the range of $X_1, \ldots, X_i$. Prove the following:

- $I(X; Z|Y) = 0$ if and only if $X \to Y \to Z$
- If $X \to Y \to Z$ then $I(X; Y) \geq I(X; Z)$, and equality holds if and only if also $X \to Z \to Y$.
- If $X \to Y \to Z \to W$ then $I(X; W) \leq I(Y; Z)$.

Question 4. Show that Fano's inequality is sharp. That is, find a jointly distributed X and Y such that you can show that the minimum of p_e over all possible $g : \mathcal{Y} \to \mathcal{X}$ satisfies

$$h(p_e) + p_e \log(|\mathcal{X}| - 1) = \mathbb{H}(X|Y).$$

(Ideally an example that works for any $|\mathcal{X}|$)

Question 5. Suppose we have two channels $p_{Y|X}$ and $p_{V|U}$ with capacities C_1 and C_2 respectively.

Suppose we form a new channel $p_{(Y,V)|(X,U)}$ by sending a pair of messages along both channels simultaneously, that is, such that

$$p_{(Y,V)|(X,U)}((y, v)|(x, u)) = p_{Y|X}(y|x)p_{V|U}(v|u).$$

What is the capacity of this combined channel?

Question 6. Suppose X and X' are discrete random variables which are independent and identically distributed.

Show that $\mathbb{P}(X = X') \geq 2^{-\mathbb{H}(X)}$.