Discrete Entropy
Exercise Sheet 2

Question 1. Suppose we have a function $f : [0, 1] \to \mathbb{R}$ satisfying the six heuristic properties we gave for the entropy function (p.6 of the notes). Show that $f(p) = \log \frac{1}{p}$.

Question 2. In the correspondence between subsets and random variables in Section 2:

- When are the variables X_A and X_B independent?
- When is X_A determined by X_B?
- When are X_A and X_B conditionally independent relative to X_C?
- What is $I(X_A, X_B|X_C)$?

(*) Can you give an inequality between cardinalities of sets where the corresponding entropy inequality doesn't hold?

(Hint: Consider X, Y and Z where each pair is independent, but determines the third)

Question 3. Let X, Y, Z be discrete random variables. Show that $H(X|Y, Z) \leq H(X|Y)$.

Suppose further that Y determines Z. Show that $H(X|Y) = H(X|Y, Z)$.

Question 4. Let X, Y and Z be discrete random variables. Then

$$H(X, Y|Z) = H(X|Z) + H(Y|X, Z).$$

Question 5. Suppose X, Y, Z and W are discrete random variables. Show that

$$I(X, Y; Z|W) = I(X; Z|W) + I(Y; Z|X, W).$$

Show that conditioning does not always decrease mutual information, that is find X, Y, Z such that'

$$I(X; Y|Z) > I(X; Y).$$

Question 6. Let V be a finite dimensional vector space over a finite field $\mathbb{F}$ and let X be a random variable which is uniformly distribution over the set of linear functionals from S to $\mathbb{F}$. Given a subspace U of V, let X_U be the restriction of the function X to the subspace U.

- Show that X_U is uniformly distributed over the linear functionals from U to $\mathbb{F}$;
- Compute $\mathbb{H}(X_U)$;
- Develop a correspondence between subspaces and random variables as in Question 2;
- (*) Are there inequalities between the dimensions of subspaces where the corresponding entropy inequality doesn't hold?