Discrete Entropy
Exercise Sheet 10

Question 1. Let H, E, V, t, F_i and w_i be as in the Weighted Shearer's Lemma. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a vector of non-negative weights such that

$$\sum_{v \in F_i} \alpha_i \geq 1$$

for each $v \in V$ (i.e α is a 'fractional cover' of the hypergraph whose edges are the F_i). Show, using the Weighted Shearer's Lemma (or directly) that,

$$\sum_{e \in E} \prod_{i=1}^{r} w_i(e_i) \leq \prod_{i=1}^{r} \left(\sum_{e_i \in E_i} w_i(e_i)^{\frac{1}{\alpha_i}} \right)^{\alpha_i}.$$

Question 2. Let $a_1, \ldots, a_n \in \mathbb{R}^+$ and let $s < r \in \mathbb{N}$ show that

$$\left(\frac{\sum_k a_k^s}{n} \right)^{\frac{1}{s}} \leq \left(\frac{\sum_k a_k^r}{n} \right)^{\frac{1}{r}}.$$

Deduce that the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$f(p) := \left(\frac{\sum_k a_k^p}{n} \right)^{\frac{1}{p}}$$

is increasing.

Using the above prove the arithmetic-geometric mean inequality.

Question 3. Following the notation of Lemma 4.28 (The weighted Shearer's Lemma) let us define for each $e \in E$

$$w(e) = \prod_{i=1}^{r} w_i(e_i).$$

Show that a necessary condition for equality to hold is that for each i and each $e^* \in E_i$

$$\frac{w_i(e^*)^t}{\sum_{e_i \in E_i} w_i(e_i)^t} = \frac{\sum_{e' \in E:\, e'_i = e^*} w(e')}{\sum_{e \in E} w(e)}$$

Question 4. Let G be a group and $G_1, \ldots, G_n$ be subgroups of G. Suppose $A \subseteq [n]$, show that $G_A = \bigcap_{i \in A} G_i$ is a subgroup of G.

Given a collection of group elements $a_i \in G$ for $1 \leq i \leq n$, show that the intersection of the cosets $\bigcap_{i \in A} a_i G_i$ is either a coset of G_A or empty.

Let us define a family of random variables as follows: Let Y be a random variable uniformly distributed on G, and let $X_i = YG_i$ for each $i \in [n]$. Given a subset $A \subseteq [n]$ calculate $\mathbb{H}(X_A)$.

Question 5. Express the inequality $I(X_1; X_2 | X_3)$ in terms of joint entropies of X_1, X_2 and X_3. Using the previous question show that for any group G and subgroups G_1, G_2, G_3

$$|G_{12}||G_{13}| \leq |G_3||G_{123}|$$

(**Prove the above directly.)