Discrete Entropy
Exercise Sheet 1

Question 1. Give an example of a set of events $\{A_i : i \in I\}$ which are pairwise independent, but not mutually independent.

Give an example of a collection of random variables $\{X_i : i \in I\}$ which are pairwise independent, but not mutually independent.

Give an example of two random variables X and Y such that $\mathbb{E}(XY) \neq \mathbb{E}(X)\mathbb{E}(Y)$.

Question 2. Let $k \geq 3$, $n \leq 2^{k/2-1}$ and let K_n be the complete graph on n vertices. Show that there is a 2-colouring $c : E(K_n) \to \{1, 2\}$ of the edges of K_n such that there is no subset $X \subseteq V(K_n)$ of size $|X| \geq k$ on which c is monochromatic.

Question 3. Let $n \geq 4$ and let $H = (V, E)$ be an n-uniform hypergraph with at most $4^{n-1}/3^n$ edges. Show that there is a 4-colouring $c : V \to [4]$ of the vertices of H such that each colour appears in every edge.

Question 4 (Kraft's inequality)**.** Let $\mathcal{F} \subset \{0, 1\}^{<\omega}$ be a finite collection of binary strings of finite length such that no member of $\mathcal{F}$ is a prefix of another. Let $n_i = |\mathcal{F} \cap \{0, 1\}^i|$ be the number of strings of length i in $\mathcal{F}$. Show that

$$\sum_i \frac{n_i}{2^i} \leq 1. \tag{1}$$

(Hint: Pick a random string of length longer than the longest in $\mathcal{F}$ and consider the probability that a string in F is a prefix of it)

Conversely, show that if (1) holds for a sequence of numbers n_i then there exists such a collection $\mathcal{F}$.

Question 5. Let $T = (x_1, x_2, \ldots, x_m)$ be a sequence of not necessarily distinct reals and let

$$T_b = \{(x_i, x_j)\, |x_i - x_j| \leq b\}.$$

a) Show that $|T_2| < 3|T_1|$

b) Prove that for every two independent identically distributed real random variables X and Y,

$$\mathbb{P}(|X - Y| \leq 2) \leq 3\mathbb{P}(|X - Y| \leq 1).$$