DGLAs & Deformation Functors

Arndt H. Ziegler

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Contents

1 Introduction

The preceding talks presented examples for deformation problems and introduced a framework to talk about them in a unified manner. We learned about deformations of associative algebras and complex structures and about the classical approach of deformation theory, which aims to associate to each deformation problem a deformation functor. However, a lot of information is lost by this procedure. Instead, we can first associate to a deformation problem a differential graded Lie algebra (dgla) (in general only unique up to quasiisomorphism). The dgla contains more structure and preserves more information about the initial deformation problem. From there there is a well defined and functorial procedure to associate to each dgla a deformation functor, whose description is today's task.
The slogan is: Over a field of characteristic zero, every deformation problem is governed by a dgla via solutions of the Maurer-Cartan equation modulo gauge action. - [Man99]

2 DGLAs and Maurer-Cartan Elements

Let's fix some notation: We work over a field k of characteristic zero. Tensor products are always taken over $k : \otimes = \otimes_k$.

This section introduces dglas and auxiliary structures, as well as the Maurer-Cartan equation, which is central to constructing deformation functors.

Definition 2.0.1. A *differential graded vector space* (dg-vector space) is a cochain complex in k-vector spaces. That is, it is a $\mathbb{Z}$-graded vector space $V = \oplus_{i\in\mathbb{Z}} V^i$ together with a linear map $d : V \to V$ of degree $+1$, i.e. $d(V^i) \subset V^{i+1}$, called differential, that satisfies $d^2 = 0$.
A morphism of dg-vector spaces $f : V \to W$ is a cochain map, i.e. a linear map of degree 0 that satisfies $d_W f = f d_V$
Assembled together they form the category of dg-vector spaces, denoted DG.

Definition 2.0.2. DG is equipped with a monoidal structure, given by tensor product of dg-vector spaces, which is the usual tensor product of cochain complexes. For $V, W \in \mathrm{DG}$, the tensor product $V \otimes W$ is defined as follows:

$$(V \otimes W)^n = \oplus_{p+q=n} V^p \otimes W^q$$

$$d_{V\otimes W}(v \otimes w) = d_V v \otimes w + (-1)^{\bar{v}} \otimes d_W w$$

Definition 2.0.3. A *commutative differential graded algebra* (cdga) is a dg-vector space (A, d) together with a morphism

$$A \otimes A \to A$$

$$a \otimes b \mapsto ab$$

called product, that satisfies

1. Associativity: $(ab)c = a(bc)$
2. Graded commutativity: $ab = (-1)^{\bar{a}\bar{b}} ba$
3. Graded Leibniz: $d(ab) = (da)b + (-1)^{\bar{a}} a(db)$

Example 2.0.1.

- cdgas are just monoids in $\mathrm{Ch}(\mathrm{Vect}_k)$
- Commutative algebras are the same as cdgas concentrated in degree zero and vice versa
- The de Rham complex $\Omega^*(M)$ of a smooth manifold M with wedge product $\wedge$ is a cdga
- Denote the de Rham complex of algebraic differential forms on the affine line by $k[t, dt]$. The underlying dg-vector space is concentrated in degree zero and one:

  $$k[t] \oplus k[t]dt$$

  The differential on a general element $p(t) + q(t)dt$ is defined as

  $$d(p(t) + q(t)dt) = \dot{p}(t)dt$$

  Multiplication is multiplication of polynomials. There are evaluation maps

  $$e_s : k[t, dt] \to k$$

  $$p(t) + q(t)dt \mapsto k(s), s \in k$$

Definition 2.0.4. A *differential graded Lie algebra* (dgla) is a dg-vector space (L, d) together with a bilinear bracket $[-,-] : L \times L \to L$ satisfying

1. Graded skewsymmetry: $[a, b] + (-1)^{\bar{a}\bar{b}}[b, a] = 0$
2. Graded Jacobi: $[a, [b, c]] = [[a, b], c] + (-1)[b, [a, c]]$
3. Graded Leibniz: $d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$

A morphism $f : L \to L'$ of dglas is a morphism of dg-vector spaces that commutes with brackets, i.e. $f([l, k]_L) = [f(l), f(k)]_{L'}$. They assemble into a category called DGLA.

Remark 2.0.1. Due to the grading of the bracket, $[a, a] = 0$ is only true for $a \in L$ even. For $a \in L$ odd, it holds that $[a, [a, a]] = 0$.

Example 2.0.2.
- Lie algebras are dglas concentrated in degree 0 and vice versa
- As introduced in talk 2, the dgla that governs the deformation theory of associative algebras, is the Hochschild cocomplex with the Gerstenhaber bracket
- The Kodaira-Spencer dgla of a compact, complex manifold X, defined by

$$KS(X)^p = \Gamma(X, \mathcal{A}^{0,p}(T_M))$$

  with Dolbeaut differential $\bar{\partial}$ and bracket given by wedge of forms and bracket of vector fields.

We can tensor cdgas with dglas:

Lemma 2.0.1. *Let L : dgla, A : cdga, $x, y \in L$, $a, b \in A$. The tensor product of dg-vector spaces $L \otimes A$ equipped with the bracket*

$$[x \otimes a, y \otimes b] = (-1)^{\bar{a}\bar{y}}[x, y] \otimes ab$$

is a dlga and the tensor product is functorial in both arguments.

Example 2.0.3.
- Let L : dgla. Then

$$(L \otimes k[t, dt])^n = L^n \otimes k[t] \oplus L^{n-1} \otimes k[t]dt$$

  Think of elements as polynomials with values in L:

$$l_n \otimes p(t) + l_{n-1} \otimes q(t)dt =: l_n(t) + l_{n-1}(t)dt$$

- Let M: smooth manifold, $\mathfrak{g}$: Lie algebra. Then $\Omega^*(M) \otimes \mathfrak{g}$ is a dgla.

After this preparatory work, we can define

Definition 2.0.5. The *Maurer-Cartan equation* of a dgla L is

$$da + \frac{1}{2}[a, a] = 0$$

for $a \in L^1$. Solutions to the Maurer-Cartan equation are called Maurer-Cartan elements of L. They assemble into a set denoted $\mathrm{MC}(L) \subset L^1$.

Lemma 2.0.2. *Morphisms of dlgas $f : L \to L'$ commute with the Maurer-Cartan equation, i.e. $f(MC(L)) \subset MC(L')$*

Proof. Let $a \in L^1$, so that $d_L a + \frac{1}{2}[a, a]_L = 0$. Then $f(d_L a + \frac{1}{2}[a, a]_L) = 0$. The LHS can be expanded:

$$f(d_L a + \frac{1}{2}[a, a]_L) = f(d_L a) + \frac{1}{2} f([a, a]_L) = d_{L'} f(a) + \frac{1}{2}[f(a), f(a)]_{L'}$$

which shows that $f(a)$ is a Maurer-Cartan element in L'. □

Example 2.0.4. Gauge theory on a trivial $\mathcal{G}$-bundle: Connections on $M \times \mathcal{G}$ are in one-to-one correspondence with elements of $\Omega^1(M; \mathfrak{g})$. The curvature/field strength of $A \in \Omega^1(M; \mathfrak{g})$ is defined by the Maurer-Cartan equation

$$F_A = dA + \frac{1}{2}[A, A]$$

It follows that flat connections on $M \times \mathcal{G}$ are in bijection with $\mathrm{MC}(\Omega^1(M) \otimes \mathfrak{g})$)

3 The Maurer-Cartan and Gauge Group Functor

Now we define the two functors that make up the deformation functor associated to a dgla.

Warning: we need the change the category Art_k from last talk: Its objects are now *commutative* local Artinian k-algebras with residue field k. Otherwise the tensor product $L \otimes \mathfrak{m}_A$ would not be defined.

Definition 3.0.1. Let L be a dgla. The Maurer-Cartan functor of L is defined as

$$\mathrm{MC}_L : Art_k \to Set$$

$$\mathrm{MC}_L(A) = \mathrm{MC}(L \otimes \mathfrak{m}_A)$$

Remark 3.0.1. MC_L is well defined. $\mathfrak{m}_A$ is the maximal ideal of the *commutative* Artinian algebra A and as such a cdga concentrated in degree zero. By 2.0.1 $L \otimes \mathfrak{m}_A$ is a dgla. Since morphisms of Artin algebras preserve the maximal ideal, $L \otimes \mathfrak{m}_A$ is functorial in A by 2.0.1: Let $f : A \to A'$, then

$$L \otimes \mathfrak{m}_A \xrightarrow{L \otimes f} L \otimes \mathfrak{m}_{A'}$$

is a morphism of dglas. By 2.0.2, this means that

$$\mathrm{MC}_L(A) \xrightarrow{\mathrm{MC}_L(f)} \mathrm{MC}_L(A)$$

is a morphism of sets. By the same reasoning, $\mathrm{MC}_L(A)$ is functorial in L.

Remark 3.0.2. Recall from talk 2 that the exponential $exp(\mathfrak{g})$ of a nilpotent Lie algebra $\mathfrak{g}$ can be viewed as a group with the same underlying set as $\mathfrak{g}$ equipped with the Baker-Campell-Hausdorff product

$$x \bullet y = x + y + \frac{1}{2}[x, y] + ...$$

Since $\mathfrak{g}$ is nilpotent, the BCH-product is a finite sum and thus well-defined. The maximal ideal $\mathfrak{m}_A$ of an Artin algebra is nilpotent. By 2.0.1, $L \otimes \mathfrak{m}_A$ is a nilpotent dgla. In particular, $L^0 \otimes \mathfrak{m}_A$ is a nilpotent Lie algebra.

Definition 3.0.2. Let L be a nilpotent dgla. The gauge action of $exp(L^0)$ on L^1 is defined as:

$$e^a * x = x + \sum_{n \geq 0} \frac{(\mathrm{ad}a)^n}{(n+1)!}([a, x] - da)$$

where $x \in L^1$, $a \in L^0$.

Lemma 3.0.1. *The set of Maurer-Cartan elements is stable under gauge action.*

Proof. see [Man], Lemma 7.5.3 □

Definition 3.0.3. Let L be a dgla. Define the gauge group functor

$$exp_L : Art_k \to Grp$$

$$exp_L(A) = exp(L^0 \otimes \mathfrak{m}_A)$$

By 3.0.2, this construction is well-defined.

4 Deformation Functors from DGLAs

Definition 4.0.1. Define the deformation functor associated to a dgla L as

$$\mathrm{Def}_L : Art_k \to Set$$

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{exp_L(A)}$$

Let's look briefly at some of the properties of these functors. We use axioms as introduced by Manetti in chapter 3 of [Man06], which differ from the classical ones of Schlessinger.

Lemma 4.0.1. *MC_L and exp_L are local in the sense of talk 4, i.e. they map k to the one-point set.*

Proof. As Artin algebra k has the maximal ideal 0.

$$\mathrm{MC}_L(k) = \mathrm{MC}_L(L \otimes 0) = \{0\} \cong \{*\}$$

$$exp_L(k) = exp(L^0 \otimes 0) = \{e^0\} \cong \{*\}$$

□

Definition 4.0.2. A local functor $F : Art_k \to Set$ is homogenous if $B \to A$ surjective implies $\eta : F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is an isomorphism.

Definition 4.0.3. A local functor $F : Art_k \to Set$ is called deformation functor, using the same setting as in the definition of homogeneity, if $B \to A$ surjective implies η is surjective and if $A = k$ implies η is an isomorphism.

Remark 4.0.1. Homogeneity implies deformation functor.

Proposition 4.0.1. *Both MC_L and exp_L are homogenous deformation functors. Def_L is a deformation functor.*

Proof. see [Man] section 7.6. □

Let's do a concrete calculation.

Proposition 4.0.2. *The tangent space of a deformation functor Def_L is*

$$T\ Def_L = Def_L(k\epsilon) = H^1(L) \otimes k\epsilon \cong H^1(L)$$

Proof.

$$\begin{aligned}\mathrm{MC}_L(k\epsilon) = \mathrm{MC}(L \otimes k\epsilon) &= \{l \otimes \alpha\epsilon \in L^1 \otimes k\epsilon | d(l \otimes \alpha\epsilon) + \frac{1}{2}[l \otimes \alpha\epsilon, l \otimes \alpha\epsilon] = 0\} \\ &= \{l \otimes \alpha\epsilon \in L^1 \otimes k\epsilon | dl \otimes \alpha\epsilon + (-1)^{\bar{l}} \frac{1}{2}[l, l] \otimes \underbrace{\alpha^2\epsilon^2}_{=0} = 0\} \\ &= Z^1(L) \otimes k\epsilon\end{aligned}$$

Let $a \in L^0 \otimes k\epsilon$, $x \in L^1 \otimes k\epsilon$.

$$e^a * x = x + \sum_{n \geq 0} \frac{(\mathrm{ad} a)^n}{(n+1)!}(\underbrace{[a, x]}_{=0} - da) = x + da + \frac{1}{2}\underbrace{[a, da]}_{=0} + ... = x - da$$

So T $\mathrm{Def}_L = \frac{Z^1 \otimes k\epsilon}{exp(L^0 \otimes k\epsilon)} = H^1(L) \otimes k\epsilon \cong H^1(L)$ □

We achieved our goal.

Proposition 4.0.3. *There is a functor*

$$DGLA \to DefFun$$

$$L \mapsto Def_L$$

where DefFun is the category of deformation functors, as defined above, and natural transformations.

Proof. As noted above, MC_L is functorial in L and so is exp_L. Together they establish the functoriality of Def_L in L. □

There is a different notion of equivalence on Maurer-Cartan elements, which is equivalent to the one induced by gauge action.

Definition 4.0.4. Let L be a dgla and $x, y \in \mathrm{MC}(L)$. We say x and y are homotopy equivalent if there exists $\xi \in \mathrm{MC}(L[t, dt])$ such that $e_0(\xi) = x$ and $e_1(\xi) = y$. Denote by $\pi_0(\mathrm{MC}_*(L))$ the quotient of $\mathrm{MC}(L)$ under homotopy equivalence.

Proposition 4.0.4. *$MC_L \to \pi_0(MC_*(L))$ factors through Def_L and $Def_L \to \pi_0(MC_*(L))$ is an isomorphism of deformation functors.*

Proof. see [Man] Corollary 7.9.8 □

The notation suggests that ξ can be thought of as an edge in a simplicial set.

Remark 4.0.2. Quasiisomorphisms of dglas induce isomorphisms on deformation functors.

References

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