Models for (∞, n)-categories

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These notes were prepared for a seminar talk on 27. May, 2019 at Universität Hamburg, given as part of the *Research seminar on higher categories* overseen by Prof. Dr. Tobias Dyckerhoff. The aim is to give a brief exposition of two models for (∞, n)-categories: Θ_n-spaces, and n-fold complete Segal spaces.

The naïve perspective: enrichment

One heuristic idea of what an (∞, n)-category should be is a category enriched[1] in $(\infty, n-1)$-categories. This thinking has the advantage of mirroring the construction we perform in the strict case:

[1] We'll leave aside what we really mean by this for the time being.

$$\begin{aligned} \mathrm{Cat}_1 &:= \mathrm{Set}\text{-}\mathrm{Cat} \\ \mathrm{Cat}_2 &:= \mathrm{Cat}_1\text{-}\mathrm{Cat} \\ &\vdots \\ \mathrm{Cat}_n &:= \mathrm{Cat}_{n-1}\text{-}\mathrm{Cat} \end{aligned}$$

and so on. In many applications, however, we have to weaken our notion of enrichment as soon as we get to the case of 2-categories.

But, the process of successive enrichments works. We have the Kan model structure on Set_Δ, which gives us a good notion for what the ∞-category $\mathrm{Cat}_{(\infty,0)} = \mathcal{S}$ should be[2]. What's more, the ample appendices of *Higher Topos Theory* provide the following theorem:

[2] To some degree, this is putting the cart before the horse, because it assumes we already know what an ∞-category is.

Theorem 1. *Let* $\mathbf{C}$ *be a "nice enough"*[3] *monoidal model category, then there is a model structure on* $\mathbf{C}$-Cat *whose weak equivalences are the "homotopy essential surjective" functors such that* $F : \mathcal{B}(x, y) \to \mathcal{C}(Fx, Fy)$ *is a weak equivalence in* $\mathbf{C}$.

[3] A combinatorial monoidal model category in which every object is cofibrant, and weak equivalences are stable under filtered colimits.

This is, in fact, how we construct the model structure on Cat_Δ modeling $(\infty, 1)$-categories.

Problem 1: This construction doesn't iterate well – the hypotheses on $\mathbf{C}$ are stronger than the conclusions we can draw about $\mathbf{C}$-Cat.

Problem 2: "Categories enriched in categories enriched in ...." are a terribly messy model to work with.

The first of these problems is not insurmountable, but the second makes the method undesirable in all but the simplest cases. Notably, even when working in the non-∞ setting, we often as not have to consider bicategories rather than strict 2-categories.

The less-naïve perspective: internalization

Both of the problems with enriching categories in $(\infty, n-1)$-categories come from the superfluous strictness of the enrichment process. We can try and relax this requirement by thinking of categories *internal* to (∞, n)-categories.

Definition 2. Let $\mathcal{C}$ be a 1-category with enough limits. A *category internal to* $\mathcal{C}$ consists of a functor

$$X : \Delta^{\mathrm{op}} \to \mathcal{C}$$

such that the canonical map

$$X_n \to \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{\times n} \tag{1}$$

is an isomorphism[4].

[4] In this 1-categorical setting, this is *far* too much data. Indeed, all one really needs are objects X_0 and X_1, units $\sigma_0 : X_0 \to X_1$, source and target maps $\partial_1, \partial_0 : X_1 \to X_0$, and a composition $X_1 \times_{X_0} X_1 \to X_1$ satisfying the usual unitality and associativity relations. These data are enough to uniquely determine the simplicial object X up to isomorphism. We think of X_0 as the "object of objects" in the internal category, and X_1 as the "object of morphisms."

Example 3. If we take $\mathcal{C} = \mathrm{Set}$, then the condition eq. (1) is nothing more or less than the requirement that the simplicial set X be the nerve of a category. As a result, we can identify categories internal to Set with 1-categories.

Example 4 (Motivation)**.** Suppose we have a category $\mathbf{C}$ internal to Grpd. Then $\mathbf{C}$ consists of two groupoids, $\mathcal{C}_0$ and $\mathcal{C}_1$, together with maps

$$\mathcal{C}_1 \underset{t}{\overset{s}{\underset{\leftarrow \mathrm{id} -}{\rightrightarrows}}} \mathcal{C}_0$$

and

$$m : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1.$$

Satisfying compatibility conditions. Applying the forgetful functor to Set which forgets the morphisms, we get a category $J(\mathbf{C})$ with

- objects: $\mathrm{Obj}(\mathcal{C}_0)$
- morphisms: $\mathrm{Obj}(\mathcal{C}_1)$.

We might reasonably ask if we can go the other way, and construct an internal category in Grpd from a category $\mathcal{D}$. The answer is yes, and in fact, there are multiple ways to do so:

1. We can simply pass through the inclusion $i : \mathrm{Set} \to \mathrm{Grpd}$ of sets as discrete groupoids to obtain an internal category $\min(\mathcal{D})$.
2. We can define $\max(\mathcal{D})$ to be the internal category with

$$\max(\mathcal{D})_0 := \mathrm{core}(\mathcal{D})$$

and

$$\max(\mathcal{D})_1 := \mathrm{core}(\mathcal{D}^{[1]}).$$

Note that

$$J(\min(\mathcal{D})) \cong \mathcal{D} \cong J(\max(\mathcal{D}_1)).$$

The upshot of this is that categories internal to groupoids provide some way of modeling 1-categories. However, there could be multiple inequivalent categories internal to groupoids that give the same 1-category. As a result, we need to provide some additional criterion to restrict which internal categories we consider. We could simply require that the category $\mathcal{C}_0$ is discrete — this actually leads to a sensible notion of $(\infty, 1)$-categories called *Segal categories*. However, we would prefer to consider the model $\max(\mathcal{D})$ from the example.

To characterize those categories internal to Grpd which come from this construction, consider the set of objects $f \in \mathrm{Obj}(\max(\mathcal{D})_1)$ corresponding to the isomorphisms in $\mathcal{D}$. Denote the full subcategory of $\max(\mathcal{D})_1$ on these objects by $\max(\mathcal{D})_E$, and note that $\mathrm{id} : \max(\mathcal{D})_0 \to \max(\mathcal{D})_1$ factors through $\max(\mathcal{D})_E$.

Claim 5. *The functor* $I : \max(\mathcal{D})_0 \to \max(\mathcal{D})_E$ *is an equivalence of categories.*

Proof. Note first that, if $f : x \to y$ is an isomorphism in $\mathcal{D}$, then there is a commutative square

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_y} \\ y & \xrightarrow[\mathrm{id}_y]{} & y \end{array}$$

displaying an equivalence between f and id_y. So I is essentially surjective. It is obvious that I is faithful, so we need only check that I is full. However, any morphism $\mathrm{id}_x \to \mathrm{id}_y$ in $\mathrm{core}(\mathcal{D}^{[1]})$ must be given by a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{\mathrm{id}_x} & x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ y & \xrightarrow[\mathrm{id}_y]{} & y \end{array}$$

and so, $f = g$, meaning that morphism in question is $I(f)$. □

Once we have shaken off the spurious strictness of this example, the higher-categorical analogue of this condition will be called the *completeness condition* on an internal category.

Segal spaces

We now want to use this process of internalization to define an ∞-category of (∞, n)-categories starting from an ∞-category of $(\infty, n-1)$-categories. Since we already have a good model for the ∞-category $\mathcal{S}$ of spaces, we will start here.

Definition 6. A *category object in an* ∞*-category* $\mathcal{C}$ is a simplicial object

$$X_\bullet : N(\Delta^{\mathrm{op}}) \to \mathcal{C}$$

such that the induced map

$$X_n \to X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \times_{X_{\{2\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$

is an equivalence in $\mathcal{C}$.[5]

We call a category object in $\mathcal{S}$ a *Segal space*.

[5] There are other, equivalent, formulations of the Segal condition. For instance, one can require that the squares

$$\begin{array}{ccc} X_m & \longrightarrow & X_k \\ \downarrow & & \downarrow \\ X_{m-k} & \longrightarrow & X_0 \end{array}$$

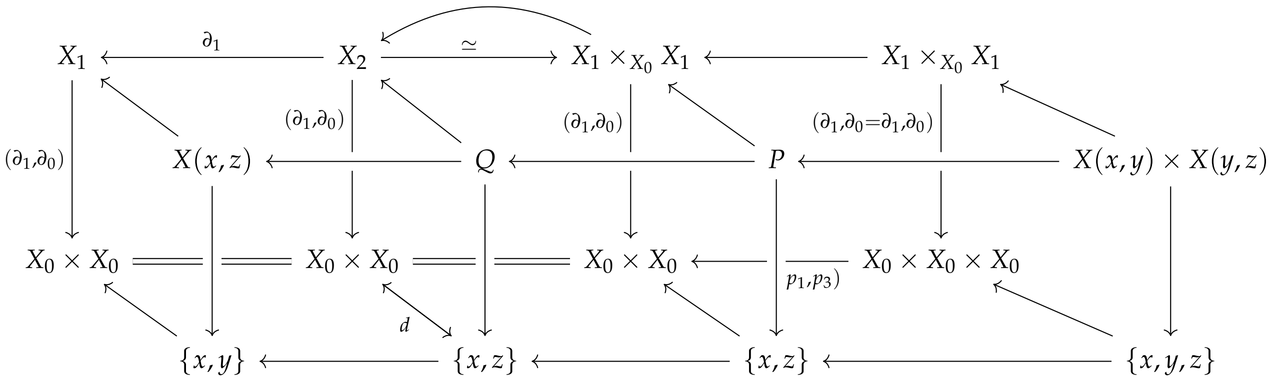
are pullback in $\mathcal{S}$.

Given a category object $X : \Delta^{\mathrm{op}} \to \mathcal{S}$, we can extract a homotopy 1-category $\mathrm{Ho}(X)$ as follows.

Construction 7. We set $\mathrm{Obj}(\mathrm{Ho}(X)) := X_{0,0}$, the 0-simplices of the image of 0^{th} space of X. For a pair of objects $x, y \in X_{0,0}$, we define a *mapping space* $X(x,y)$ to be the (∞-categorical) pullback.

$$\begin{array}{ccc} X(x,y) & \longrightarrow & X_1 \\ \downarrow & & \downarrow {\scriptstyle (\partial_1,\partial_0)} \\ \{x,y\} & \longrightarrow & X_0 \times X_0 \end{array}$$

Choosing an inverse to the equivalence $X_2 \to X_1 \times_{X_0} X_1$ (unique up to contractible choice), we obtain:



Where the 'transverse slice' squares are pullback. We call the induced morphism $X(x,y) \times X(y,z) \to X(x,y)$ the composition. The Segal conditions imply that it will be associative and unital up to homotopy.

The homotopy category $\mathrm{Ho}(X)$ then has hom-sets $\pi_0 X(x,y)$. We call the elements $f \in X(x,y)$ that become isomorphisms in $\mathrm{Ho}(X)$ the *equivalences* of X, and denote the full sub- Kan complex of X_1 on the equivalences by X_{WE}. Note that, since every identity is an equivalence, the canonical morphism

$$s_0 : X_0 \to X_1$$

factors through X_{WE}

Definition 8. A Segal space is called *complete* if the canonical morphism $X_0 \to X_{WE}$ is an equivalence.

Example 9 (Quasi-categories and complete Segal spaces)**.** Let $X : \Delta^{\mathrm{op}} \to \mathcal{S}$ be a complete Segal space. We can view X as a bisimplicial set $X : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Set}$. Then we can extract a simplicial set

$X_{n,0} : \Delta^{op} \to \mathrm{Set}$ from X. By work of Joyal-Tierney, this is, in fact a quasi-category.

In the other direction, we do not have to directly resort to simplicial sets. Let $X \in \mathrm{Cat}_{(\infty,1)}$ and consider the cosimplicial object

$$\Delta \to \mathrm{Cat}_{(\infty,1)}, \quad [n] \mapsto \Delta^n.$$

We can then define the composite:

$$\Delta^{op} \to \mathrm{Cat}^{op}_{(\infty,1)} \xrightarrow{\mathcal{Y}_X} \mathrm{Cat}_{(\infty,1)} \xrightarrow{(-)^{\simeq}} \mathcal{S}.$$

Which sends $[n] \mapsto \mathrm{Fun}(\Delta^n, X)^{\simeq}$. This is a complete Segal space.[6] We denote this composite by $CN(X)$.

[6] See Joyal-Tierney

Theorem 1 (Joyal-Tierney). *The full sub-∞-category* $\mathrm{CSS}_1 \subset \mathrm{Fun}(\Delta^{op}, \mathcal{S})$ *on the complete Segal spaces is equivalent to* $\mathrm{Cat}_{(\infty,1)}$.

To identity the sub-∞-category of spaces in the higher Segal spaces, we take the following intuition: In a complete Segal space X, the space X_0 is the underlying ∞-groupoid of X, i.e. the 'space of objects'[7]. As a result, a complete Segal space representing an $(\infty, 0)$-category should be completely determined by X_0.

[7] In some sense, this is precisely what the completeness condition says

Definition 10. A complete Segal space is said to be *essentially constant* if the functor $X : \Delta^{op} \to \mathcal{S}$ factors through $\mathcal{S}^{\simeq}$.

Since Δ^{op} is contractible, we see that there is an equivalence of ∞-categories

$$\mathcal{S} \simeq \mathrm{CSS}_{\mathrm{EC}}$$

We thus identify $\mathcal{S} \subset \mathrm{CSS}_1$ as a full sub-∞-category.

Remark 11. Note that here, we have defined only an $(\infty, 1)$-category of complete Segal spaces rather than an $(\infty, 2)$-category. As we continue to higher categories, we will continue to follow this approach. It is, however, possible to define (∞, n)-categories of functors between (∞, n)-categories in general. See, for example, Rezk's 'A Cartesian presentation of weak n-categories'.

Remark 12 (An alternate characterization of completeness)**.** Let E be the nerve of the category which has two objects and a unique isomorphism between them. Then there is a unique morphism $E \to \Delta^0$ in $\mathrm{Cat}_{(\infty,1)}$. This gives us a morphism

$$f : CN(E) \to CN(\Delta^0),$$

of complete Segal spaces. A Segal space X is then complete if and only if the morphism

$$X_0 \simeq \mathrm{Map}(CN(\Delta^0), X) \to \mathrm{Map}(CN(E), X) \cong X_{WE}$$

induced by pulling back along f is an equivalence.

Iterating

Now that we have defined an internalization procedure for ∞-categories, we wish to iterate it to obtain models for (∞, n)-categories.

Definition 13. An *n-fold Segal space* is a category object in $(n-1)$-fold Segal spaces.

Note that an n-fold Segal space can be viewed as a functor

$$X : \Delta^{\times n} \to \mathcal{S}.$$

Definition 14. We call a n-fold Segal space X *complete* if

1. X is a category object in $(n-1)$-fold complete Segal spaces.
2. The $(n-1)$-fold simplicial space $X_0 := X_{0,\bullet,\dots,\bullet}$ is essentially constant.[8]
3. The simplicial space $Y_\bullet := X_{\bullet,0,\dots,0}$ is a complete Segal space.

We denote the $(\infty, 1)$-category of n-fold complete Segal spaces by $\mathrm{CSS}_n \subset \mathrm{Fun}(\Delta^{\times n}, \mathcal{S})$.

[8] In this context, essentially constant once again means simply that $X_0 : \Delta^{\times(n-1)} \to \mathcal{S}$ factors through $\mathcal{S}^{\simeq}$. This condition guarantees that we will have a *space* of objects.

Remark 15. It is worth briefly teasing out what, precisely condition 3. means. We can see the space $Y_\bullet := X_{\bullet,0,\dots,0}$ as the underlying $(\infty, 1)$-category of X. The third condition is therefore simply guaranteeing that it will, indeed, be an $(\infty, 1)$-category.

Theorem 16 (Barwick-Schommer-Pries)**.** *The $(\infty, 1)$-category CSS_n is a model for the ∞-category of (∞, n)-categories.*

Notation 17. When we do not wish to specify a particular model, we will write $\mathrm{Cat}_{(\infty,n)}$ for the ∞-category of (∞, n)-categories.

Interlude: Monoidal and symmetric monoidal (∞, n)-categories.

While, for the most part, the details of monoidal-ness won't come into play in the sequel, let us briefly consider what a monoidal (∞, n)-category might be.

Example 18 (Motivation)**.** A 2-category with a single object is a monoidal 1-category.

Following this definition, we might then say

Definition 19. A monoidal (∞, n)-category is an $(\infty, n+1)$-category with a contractible space of objects.

Remark 20. While this definition has the great benefit of being concise, it presents a number of problems. Most notably, it is quite difficult to see, in this formalism, how or when a monoidal structure on an (∞, n)-category can be promoted to a *symmetric* monoidal structure.

However, there is another way to define monoidal (∞, n)-categories. A monoidal structure should be a coherently associative and unital multiplication law on an (∞, n)-category $\mathcal{C}$. We can therefore make the definition

Definition 21. A *monoidal* (∞, n)*-category* is an associative algebra in the ∞-category of (∞, n)-categories. A *symmetric monoidal* (∞, n)*-category* is a commutative[9] algebra in the ∞-category of (∞, n)-categories.[10]

This definition has the advantage of encoding both monoidalness and symmetric monoidalness. All that remains to us is to try and relate the two definitions.

Theorem 22 (Lurie, HA 4.1.2.10)**.** *For every ∞-category with finite products, There is an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{Mon}}(\Delta^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{A}\mathrm{ss}}(\mathcal{C})$$

Where the ∞-category on the left is the full sub-∞-category on those functors X which are category objects with $X_0 \simeq *$.

Consequently, we have equivalences

$$\left\{\begin{smallmatrix} n\text{-fold} \\ \text{Segal spaces} \\ \text{with } X_0 \simeq * \end{smallmatrix}\right\} \simeq \mathrm{Fun}^{\mathrm{Mon}}(\Delta^{\mathrm{op}}, \mathrm{CSS}_{n-1}) \simeq \mathrm{Alg}_{\mathcal{A}\mathrm{ss}}(\mathrm{CSS}_{n-1})$$

giving us the equivalence of our two definitions.

Θ_n*-spaces*

We now turn our attention to a second model of (∞, n)-categories, that used in Ayala-Francis-Rozenblyum.

Definition 23. Let $\mathcal{C}$ be a 1-category. We define the wreath product

$$\Delta \wr \mathcal{C}$$

to have objects given by $([n], (c_1, \ldots, c_n))$, $c_i \in \mathcal{C}$, and morphisms $([n], (c_1, \ldots, c_n)) \to ([m], (d_1, \ldots, d_n))$ given by

- A morphism $\phi : [n] \to [m]$ in Δ.
- For every $0 < i \leq n$ and every $\phi(i-1) < j \leq \phi(i)$, a morphism

  $$f_{i,j} : c_i \to d_j$$

  in $\mathcal{C}$.

We define $\Theta_n := \Delta^{\wr n}$ to be the n-fold wreath product of Δ with itself.

Construction 24. There is a functor

$$\delta^{\mathcal{C}} : \Delta \times \mathcal{C} \to \Delta \wr \mathcal{C}$$

constructed as follows.

- On objects, we map

  $$([n], c) \mapsto ([n], (\underbrace{c, \ldots, c}_{\times n}))$$

[9] More precisely, $\mathbb{E}_\infty$

[10] When we say an algebra in the ∞-category of (∞, n)-categories, we mean in the Cartesian monoidal structure on this ∞-category. We have not proved, nor do we essay to prove here, that $\mathrm{Cat}_{\infty,n}$ has a sufficient supply of limits for this task. This is, however, true. See Lurie's $(\infty, 2)$*-categories and the Goodwillie Calculus I* for details.

- On morphisms, we send $([n], c) \overset{f,g}{\to} ([m], d)$ to the morphism $(f, \{g\}_{i,j})$.

Iterating this procedure and composing, we get a functor

$$\delta^n : \Delta^{\times n} \to \Theta_n.$$

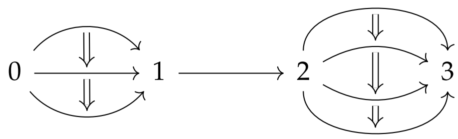
Note that this functor is not full, faithful, or essentially surjective.

We want to think of the objects in Θ_n as parameterizing 'n-categorical diagrams of a certain shape'. To get a feel for this, it is useful to restrict to the case where $n = 2$.[11]

[11] Mostly so we can draw the diagrams in question.

Example 25. Consider the object $([3], ([2], [0], [3])) \in \Theta_2$. We think of the first object, $[3]$, as specifying which objects we consider as well as the homotopy type of the hom-categories, i.e. we get objects $0, 1, 2, 3$, and one homotopy type of morphisms $i \to j$ for $i \leq j$.

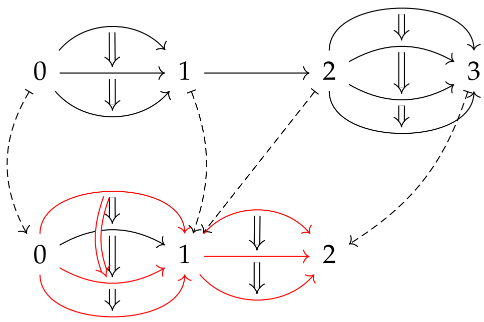
We think of the list $[2], [0], [3]$ as giving (as posets) the hom-categories. Diagrammatically, this means that our chosen object looks like



So now, what do morphisms do? Lets consider the morphism

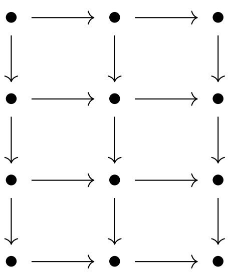
$$([3], ([2], [0], [3])) \to ([2], ([3], [2])$$

given by $\sigma_1 : [3] \to [2]$ and $\delta_1 : [2] \to [3]$ and $\sigma_0 : [3] \to [2]$. We can visualize this as

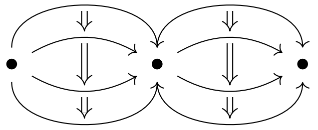


i.e. the morphism acts on the hom-categories as δ_1 and σ_0.

Remark 26. The functor δ^2 admits a nice pictorial intuition. An object in Δ^2, can be viewed as a grid. The functor δ^2 merely collapses this grid. Take, for example, the object $([2], [3])$. Then we have that the grid



is sent to the diagram



Definition 27. A Θ_n*-space* is a functor $\Theta_n^{\mathrm{op}} \to \mathcal{S}$.

We would like to impose conditions on Θ_n-spaces analogous to the Segal conditions, so that we have a natural notion of *composition/pasting* of diagrams. To see how this can be achieved, we reformulate the Segal conditions:

Definition 28. The *inert subcategory* $\Delta_{\mathrm{in}} \subset \Delta$ contains all objects of Δ, but only those morphisms which are the inclusions of subintervals. A colimit diagram $F : [1] \times [1] \to \Delta$, visualized as

$$\begin{array}{ccc} F_{00} & \longrightarrow & F_{01} \\ \downarrow & & \downarrow \\ F_{10} & \longrightarrow & F_{11} \end{array}$$

is called a *Segal diagram* if it factors through Δ_{in}.

Claim 29. *A simplicial space $X : \Delta^{\mathrm{op}} \to \mathcal{S}$ is a Segal space if and only if it sends Segal diagrams to limit diagrams.*

Proof. Since the diagrams defining the Segal conditions are Segal diagrams, 'if' follows immediately. In the other direction, the claim is the same as saying that in a Segal space the diagrams

$$\begin{array}{ccc} X_1^{\times_{X_0} n+\ell+k} & \longrightarrow & X_1^{\times_{X_0} \ell+k} \\ \downarrow & & \downarrow \\ X_1^{\times_{X_0} n+\ell} & \longrightarrow & X_1^{\times_{X_0} \ell} \end{array}$$

are pullback. □

Definition 30. We define the *inert subcategory* $\Theta_{n,\mathrm{in}} \subset \Theta_n$ inductively. We set $\Theta_{1,\mathrm{in}} := \Delta_{\mathrm{in}}$, and then define

$$\Theta_{n,\mathrm{in}} := \Delta_{\mathrm{in}} \wr \Theta_{n-1,\mathrm{in}} \subset \Theta_n.$$

A colimit diagram $[1] \times [1] \to \Theta_n$ is called a *Segal diagram* if it factors through $\Theta_{n,\mathrm{in}}$.

A Θ_n-space $X : \Theta_n^{\mathrm{op}} \to \mathcal{S}$ is called a *Segal* Θ_n*-space* if it sends all Segal diagrams to limit diagrams.

Remark 31. We note that, given a Segal Θ_n-space $X : \Theta_n^{\mathrm{op}} \to \mathcal{S}$, we get a n-fold simplicial space $(\delta^n)^*(X) : \Delta^{\times n} \to \mathcal{S}$ by pulling back along the canonical morphism

$$\delta^n : \Delta^{\times n} \to \Theta_n.$$

Since the image of an inert diagram in any factor of $\Delta^{\times n}$ is an inert diagram in Θ_n, the fact that X is Segal implies that $(\delta^n)^*(X)$ is n-fold Segal.

Completeness

Construction 32 (Pushing forward)**.** Let $\mathcal{C}$ be a 1-category, and define a functor[12]

$$T_!^{\mathcal{C}} : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}((\Delta \wr \mathcal{C})^{\mathrm{op}}, \mathcal{S})$$

Given on objects by

$$T_!^{\mathcal{C}}(X)([m], (c_1, \ldots, c_m)) = X([m]).$$

In particular, we get functors

$$T_n : \mathrm{Fun}((\Delta^{\times n})^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathcal{S})$$

given by applying this construction.

Notation 33. Let E be the nerve of the walking isomorphism, we denote by

$$f_n : E_n \to *_n$$

the image of the morphism $f : E \to \Delta^0$ under $T_n \circ CN$.

Construction 34. Define functors

$$\sigma_n : \Theta_{n-1} \to \Theta_n$$

by sending $c \mapsto ([1], (c))$.

Definition 35. We say that a Segal Θ_n-space X is *complete* if

1. The morphism

$$\mathrm{Map}(*_n, X) \to \mathrm{Map}(E_n, X)$$

   induced by f_n is an equivalence.

2. The Θ_{n-1}-space $\sigma_n^*(X)$ is complete.

This definition is, once again, inductive. We first define a Segal Θ_1-space to be complete if it is a complete Segal space, and then apply the definition above.

We denote the full subcategory of $\mathrm{Fun}(\Theta_n^{\mathrm{op}}, \mathcal{S})$ on the complete Segal Θ_n spaces by Θ_n CSS.

Theorem 36 (Barwick-Schommer-Pries, Bergner-Rezk)**.** *The functor δ^n defines an equivalence*

$$\Theta_n\,\mathrm{CSS} \simeq \mathrm{CSS}_n$$

of ∞-categories.

[12] The notation here is a modification of that used by Rezk. T is defined as the Kan extension of a functor $T : \Delta \to \mathrm{Fun}((\Delta \wr \mathcal{C})^{\mathrm{op}}, \mathcal{S})$ given by $[n] \mapsto (([m], (c_1, \ldots, c_m)) \mapsto \Delta([n], [m])$ along the Yoneda embedding.

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