Seminar on the Cobordism Hypothesis

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Abstract

The goal here is to introduce the Cobordism Hypothesis as conjectured by Baez and Dolan and then formalized by Lurie. For this, we try to understand the bordism category Bord_n as an (∞, n)-category and the notion of fully dualizable objects.

1 Introduction

Throughout these notes, we will mainly follow the constructions made in [Lur09] and [CS15]. We begin with reviewing the notion of TFTs and the statement of the Cobordism Hypothesis as formulated [BD95]. In this section, *manifolds* will always mean smooth and compact manifolds (possibly with boundary).

The oriented cobordism category $\mathrm{Cob(n)}^{\mathrm{or}}$ consists of:

- objects are (oriented) closed $(n-1)$-manifolds
- A morphism $[M] : X \to Y$ is represented by an n-bordism M, i.e. an (oriented) n-dimensional manifold M equipped with boundary parametrisation $\partial M \cong \overline{X} \amalg Y$, where $\overline{X}$ denotes the manifold X with reversed orientation. Two such bordisms M, M' are equivalent if there is a diffeomorphism $M \cong M'$ compatible with the boundary parametrisations.
- For $X \in \mathrm{Cob(n)}^{\mathrm{or}}$, the identity morphism id_X is represented by the cylinder bordism $X \times [0, 1]$.
- Composition is given by gluing[1].

This category is symmetric monoidal with tensor product given by the disjoint union of manifolds and unit by the empty set $\emptyset$ regarded as an $(n-1)$-dimensional manifold.

We begin with the classical definition of a *topological field theory* as formulated by the Atiyah-Segal axioms.

Definition 1. An (oriented) TFT of dimension n is a symmetric moinoidal functor $\mathcal{Z} : \mathrm{Cob(n)}^{\mathrm{or}} \to \mathrm{Vect}$.

[1]Since we deal with smooth manifolds, the smooth structure on the resulting glued manifold depends on some choice of smooth structure on collars. However, any two choices will give diffeomorphic structures and thus representing the same morphism

Remark. One can replace the category of oriented cobordisms $\mathrm{Cob(n)}^{\mathrm{or}}$ by the category of unoriented cobordisms Cob(n) or the category of framed[2] cobordisms $\mathrm{Cob(n)}^{\mathrm{fr}}$ and thus define unoriented or framed TFTs.

In particular, an n-dimensional TFT assigns to each $(n-1)$-dimensional closed manifold a vector space and to each bordism a linear map between the vector spaces assigned to its incoming and outgoing boundaries. In particular, to a closed n-manifold regarded as bordism $\emptyset \to \emptyset$, it assigns a linear map $\Bbbk \to \Bbbk$, i.e. a number $k \in \Bbbk$. This number is an invariant of closed n-manifolds. We proceed with the lowest dimensional examples.

Topological Field Theories of dimension $n = 1$.

The objects in Cob(1) are generated by points with positive and negative orientation. The morphisms are then represented by closed intervals and circles. By unravelling the definition, a 1-dimensional TFT consists of several data:

- Vector spaces $\mathcal{Z}(\bullet^+) =: X$ and $\mathcal{Z}(\bullet^-) =: Y$ for each orientation on the point.
- By evaluating $\mathcal{Z}$ on the closed interval regarded as a bordism $\bullet^+ \to \bullet^+$, $\bullet^+ \amalg \bullet^- \to \emptyset$ and so on, we get the identities on X and Y and maps $X \otimes Y \to \Bbbk$ and $\Bbbk \to Y \otimes X$

These maps exhibit Y as the dual vector space to X. In particular, both vector spaces are finite dimensional. Conversely, one can also construct a 1-dimensional TFT for a given finite dimensional vector space[3]. Such TFTs are in fact classified by dualizable vector spaces, i.e. finite dimensional vector spaces. One can use the above data to compute an invariant on 1-dimensional closed manifolds. For instance, the evaluation on the circle gives us the dimension of X, i.e. $\mathcal{Z}(\mathbb{S}^1) = \dim(X)$ (which we compute by breaking $\mathbb{S}^1$ in two pieces and use functoriality).

Topological Field Theories in dimension $n = 2$.

The objects in Cob(2) are 1-dimensional closed manifolds, i.e. disjoint unions of $\mathbb{S}^1$. Furtheremore, one can obtain every connected surface with boundary by gluing disks, trinions (pair of pants) and cylinders together (see Figure 1). Thus, a TFT $\mathcal{Z}$ of dimension 2 gives the following data:

- A vector space $\mathcal{Z}(\mathbb{S}^1) =: A$
- The disk can be thought as a bordism $\emptyset \to \mathbb{S}^1$ or as a bordism $\mathbb{S}^1 \to \emptyset$. The trinion can be thought as a bordism $\mathbb{S}^1 \amalg \mathbb{S}^1 \to \mathbb{S}^1$ (and in the other direction). Applying our functor $\mathcal{Z}$ gives us linear maps $\eta : \Bbbk \to A$, $\epsilon : A \to \Bbbk$, $\mu : A \otimes A \to A$ and $\Delta : A \to A \otimes A$.

It is straighforward to check that $(A, \eta, \mu, \epsilon, \Delta)$ forms a commutative Frobenius algebra. Conversely, given a commutative Frobenius algebra A, one can construct a 2-dimensional TFT with $\mathcal{Z}(\mathbb{S}^1) = A$. In fact, the evaluation on the circle gives us an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Cob}(2), \mathrm{Vect}) \simeq \mathrm{ComFrob}$$

[2] Let M be a smooth manifold of dimension $m \leq n$. An n-framing on M is a trivilisation of the vector bundle $TM \oplus \mathbb{R}^{n-m}$.

[3] Any n-dimensional TFT always takes values in finite dimensional vector spaces, which follows from its definition axioms.

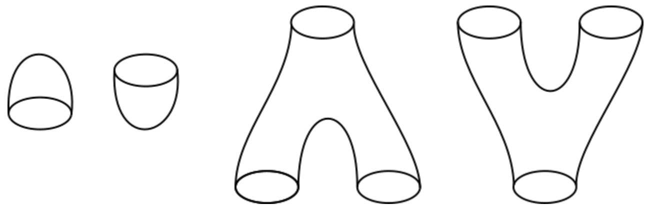
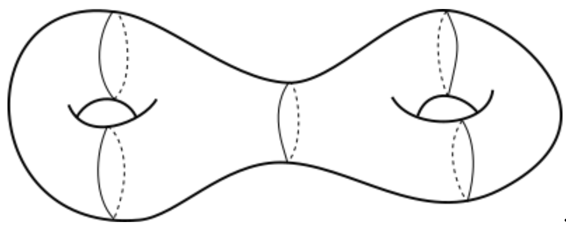


Figure 1

between the category of 2-dimensional TFTs and the groupoid of commutative Frobenius algebras (in particular $\mathrm{Fun}^{\otimes}(\mathrm{Cob}(2), \mathrm{Vect})$ is a groupoid).



Once again, one can compute an invariant on closed surfaces by breaking them into smaller pieces and using functoriality of TFTs. For instance, $\mathcal{Z}(\mathbb{S}^2) = \epsilon \circ \eta$, while $\mathcal{Z}(\mathbb{S}^1 \times \mathbb{S}^1) = \dim(A)$.

Therefore, one should not think of TFTs as invariants of closed manifolds, but rather as a set of rules on how to compute these invariants.

Extended Topological Field Theories In computing the invariants with TFTs so far, we were only able to cut along manifolds of codimension 1. Ideally, we would like to continue cutting even to lower dimensions for greater computability. This process is captured by extended TFTs.

Definition 2. Let $\mathcal{C}$ be a symmetric monoidal n-category. A (fully) extended ($\mathcal{C}$-valued) TFT of dimension n is a symmetric monoidal functor $\mathcal{Z} : \mathrm{Cob(n)}_{\mathrm{ext}} \to \mathcal{C}$.

where the n-category of n-cobordisms $\mathrm{Cob(n)}_{\mathrm{ext}}$ consists roughly of:

- Objects are points
- 1-morphisms are given by bordisms between points
- 2-morphisms are bordisms between bordisms
- ...
- $(n-1)$-morphisms are bordisms between ... between bordisms
- Isomoprhism classes of bordisms.

Composition is given by gluing and the monoidal product by taking the disjoint union. Even though it becomes increasingly harder to formulate it in a precise manner, extended TFTs are in some sense *simpler* in that they lead to more computable invariants. In the case of extended framed TFTs this idea is captured in the original statement of the Cobordism Hypothesis [BD95].

Theorem 1. *Let $\mathcal{C}$ be a symmetric monoidal n-category. The evaluation of $\mathcal{Z} \mapsto \mathcal{Z}(\bullet)$ determines a bijection between isomorphism classes of fully extended framed TFTs of dimension n and isomorphism classes of fully dualizable objects in $\mathcal{C}$.*

Remark. One could also introduce partially extended TFTs by considering for $k < n$ k-categories $\mathrm{Cob(n)}_k$ and so on.

Our next goal will be to extend the notion of n-bordisms by defining the (∞, n)-category Bord_n. Informally, one can think of this by extending $\mathrm{Cob(n)}_{\mathrm{ext}}$ up:

- ...
- n-morphisms are bordisms between bordisms ...
- $(n+1)$-morphisms are diffeomorphisms between ...
- $(n+2)$-morphisms are homotopies
- ...
- homotopies between homotopies ...

2 The (∞, n)-category of bordisms Bord_n

The model for (∞, n) used here is that of complete n-fold Segal spaces. Recall that there is process of turning any n-fold Segal space X into a complete n-fold Segal space $\hat{X}$ called the completion of X. With this in mind, we give the definition of an n-fold Segal space $\mathrm{PreBord}_n$, which is not necessarily complete. The bordism category Bord_n will then be its completion. Informally, we should think of elements in $(\mathrm{PreBord}_n)_{k_1,\dots,k_n}$ as a collection of k_i composable bordisms in the ith direction.

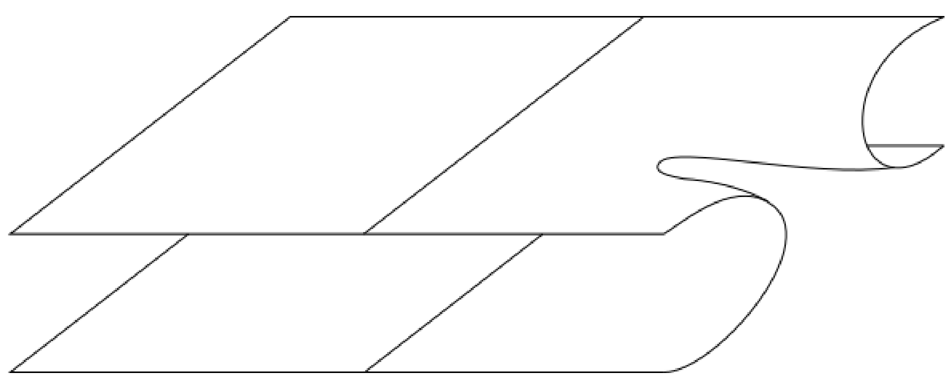


Figure 2: An element in $\mathrm{PreBord}_2$ for $k_1 = 1$ and $k_2 = 2$.

The following definition is given in [CS15]. For a comparison with the definition in [Lur09] see Appendix B

Definition 3. Let V be a finite dimensional vector space. For non-negative integers $k_1, \dots, k_n$, we define the points in $(\mathrm{PreBord}_n^V)_{k_1,\dots,k_n}$ to be tuples $(M, (t_0^i \leq \dots \leq t_{k_i}^i)_{1\leq i\leq n})$ such that

1. M is a closed n-dimensional submanifold[4] of $V \times \mathbb{R}^n$
2. The projection $\pi : M \hookrightarrow V \times \mathbb{R}^n \to \mathbb{R}^n$ is proper.
3. For every subset $S \subset \{1, \dots, n\}$ and non-negative integers $j_i \leq k_i$ for each $i \in S$, consider the map $p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$. At every point $x \in p_{\{i\}}^{-1}(\{t_0^i, \dots, t_{k_i}^i\})$ the map $p_{\{i,\dots,n\}}$ is submersive.

Motivated by our informal description of $(\mathrm{PreBord}_n)_{k_1,\dots,k_n}$, we imagine that the tuple $(M, (t_0^i \leq \dots \leq t_{k_i}^i)_{1\leq i\leq n})$ represents a collection of k_i-composable n-bordisms in the ith direction which are obtained by cutting M along the points t_j^i. Thus, the ordered tuples $t_0^i \leq \dots \leq t_{k_i}^i$ should be thought of as cut points in each time direction. The set $(\mathrm{PreBord}_n^V)_{k_1,\dots,k_n}$ will be endowed with topology induced by the Whitney C^∞-topology on the space of embeddings $\mathrm{Emb}(M, V \times \mathbb{R}^n)$.

Let $\mathrm{Sub}(V \times \mathbb{R}^n)$ denote the set of closed n-dimensional submanifolds $M \subset V \times \mathrm{R}^n$. Moreover, the space of smooth embeddings $\mathrm{Emb}(M, V \times \mathbb{R}^n)$ carries an action of the group of diffeomorphisms $\mathrm{Diff}(M)$. This leads to the identifaction

$$\coprod_{[M]} \mathrm{Emb}(M, V \times \mathbb{R}^n)/\mathrm{Diff}(M) \cong \mathrm{Sub}(V \times \mathrm{R}^n)$$

which endows $\mathrm{Sub}(V \times \mathbb{R}^n)$ with the quotient topology. More on this topology can be found in [Gal11].

Given a morphism $(f_i : [k_i'] \to [k_i])_{1\leq i\leq n}$ in $\Delta^{\times n}$, we define a function

$$(\mathrm{PreBord}_n^V)_{k_1,\dots,k_n} \to (\mathrm{PreBord}_n^V)_{k_1',\dots,k_n'}$$

$$(M, (t_0^i \leq \dots \leq t_{k_i}^i)_{1\leq i\leq n}) \to (M, (t_{f_i(0)}^i \leq \dots \leq t_{f_i(k_i)}^i)_{1\leq i\leq n}).$$

In this way $\mathrm{PreBord}_n^V$ defines an n-fold simplicial space.

Proposition 2.1. *This forms an n-fold Segal space* $(\mathrm{PreBord}_n^V)_{\bullet,\dots,\bullet}$.

Fix an infinite dimensional vector space $\mathbb{R}^\infty$ of countable dimension. Define

$$\mathrm{PreBord}_n := \lim_{V\subset\mathbb{R}^\infty} \mathrm{PreBord}_n^V$$

as the direct limit over all finite dimensional sub-vector spaces in $\mathbb{R}^\infty$ (any different choice of infinite dimensional vector space leads to equivalent objects).

Remark. In [CS15] a finer variant of $\mathrm{Bord}_n^{\mathrm{int}}$ was introduced by replacing the cut points $t_0^i \leq \dots t_{k_i}^i$ in Definition 3 by ordered intervals $I_0^i \leq I_{k_i}^i$. In this framework, we imagine elements in $\mathrm{PreBord}_n^{\mathrm{int}}$ to represent composable bordisms with collars. There is an obvious map $\mathrm{PreBord}_n^{\mathrm{int}} \to \mathrm{PreBord}_n$ by taking the middle point of each interval as a cut point. This is in fact a weak equivalence. The proof of Proposition 2.1 is then given only for this interval variant.

[4]In this section a manifold can be non-compact.

3 Fully Dualizable Objects

When considering (oriented) TFTs of dimension 1, we found a classification by restricting to finite dimensional vector spaces (i.e. which admit duals). This result can generalized for any target category. Let $\mathcal{C}$ be a symmetric monoidal category. Then, the evaluation $\mathcal{Z} \mapsto \mathcal{Z}(\bullet)$ induces an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Cob}(1), \mathcal{C}) \simeq \mathrm{core}(\mathcal{C}^{\mathrm{d}}),$$

where $\mathcal{C}^{\mathrm{d}}$ denotes the full subcategory of dualizable objects. We say that X in $\mathcal{C}$ admits *right dual* if there exists an X^* together with evaluation map

$$ev_X : X \otimes X^* \to \mathbb{1}$$

and coevaluation

$$coev_X : \mathbb{1} \to X^* \otimes X$$

such that they satisfy the (rigidity) axioms

$$\mathrm{id}_X = X \simeq X \otimes \mathbb{1} \xrightarrow{\mathrm{id}_X \otimes coev_X} X \otimes (X^* \otimes X) \simeq (X \otimes X^*) \otimes X \xrightarrow{ev_X \otimes \mathrm{id}_X} \mathbb{1} \otimes X$$

and

$$\mathrm{id}_{X^*} = X^* \simeq \mathbb{1} \otimes X^* \xrightarrow{coev_X \otimes \mathrm{id}_{X^*}} (X^* \otimes X) \otimes \simeq X^* \otimes (X \otimes X^*) \xrightarrow{\mathrm{id}_{X^*} \otimes ev_X} X^* \otimes \mathbb{1} \simeq X^*.$$

In a similar way (by inverting the order on tensor products), we say that X admits a left dual. If an object has a left and a right dual, we call it *dualizable*. We say that $\mathcal{C}$ has duals if every object has a dual.

To any monoidal category $\mathcal{C}$, one can associate a 2-category $\mathcal{BC}$ which has one object $\star$ and $\mathrm{Map}_{\mathcal{BC}}(\star, \star) = \mathcal{C}$ where composition is given by the tensor product and the identity by the unit in $\mathcal{C}$. Then, a symmetric monoidal category $\mathcal{C}$ has duals if and only if $\mathcal{BC}$ has adjoints.

Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Recall that its homotopy 1-category $h_1(\mathcal{C})$ inherits a symmetric monoidal structure. We say that $\mathcal{C}$ has *duals for objects* if $h_1(\mathcal{C})$ has duals.

Let $\mathcal{C}$ be an $(\infty, n \geq 2)$-category. We say that $\mathcal{C}$ has *adjoints for 1-morphisms* if its homotopy 2-category $h_2(\mathcal{C})$ has adjoints. Inductively, we say that $\mathcal{C}$ has adjoints for k-morphisms for $k < n$ if for all pair of objects $X, Y \in \mathcal{C}$ the $(\infty, n-1)$-category $\mathrm{Map}(X, Y)$ has adjoints for $(k-1)$-morphisms. We say that $\mathcal{C}$ *admits adjoints* if it has adjoints for all k-morphisms where $1 < k < n$.

Definition 4. A monoidal (∞, n)-category $\mathcal{C}$ *admits duals* if it has duals for objects and adjoints.

Similarly to the ordinary categories, given a monoidal (∞, n)-category, we associate an $(\infty, n+1)$-category $\mathcal{BC}$ (with essentially one object). Then, the condition that $\mathcal{C}$ admits duals is equivalent to the condition that $\mathcal{BC}$ admits adjoints. Finally, we can define what fully dualizable objects are.

Definition 5. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then, there is a symmetric monoidal (∞, n)-category $\mathcal{C}^{\mathrm{fd}}$ with duals together with symmetric monoidal functor $\mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$ with the universal property that for any other (∞, n)-category with duals $\mathcal{D}$, there is an equivalence

$$\mathrm{Fun}^{\otimes}(\mathcal{D}, \mathcal{C}^{\mathrm{fd}}) \xrightarrow{\simeq} \mathrm{Fun}^{\otimes}(\mathcal{D}, \mathcal{C})$$

Then, we say that an object $X \in \mathcal{C}$ is *fully dualizable* if it is in the essential image of $\mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$.

Example 1. This example is given in [Sch11]. Consider the 2-category Alg_2 with:

- objects are algebras over a ring k
- 1-morphisms from A to B are B-A-bimodules
- 2-morphisms are bimodule maps
- composition is given by the tensor product of bimodules

The tensor product of algebras gives Alg_2 a symmetric monoidal structure. An equivalence in Alg_2 coincides with the notion of *Morita equivalence*. Every algebra $A \in \mathrm{Alg}_2$ is 1-dualizable with dual given by the opposite algebra A^{op}. An algebra A is fully-dualizable in this category if A is projective as an A^e-module, where $A^e = A \otimes A^{\mathrm{op}}$ is the enveloping algebra over A, and projective as an k-module.

Example 2. For dualizability in the 3-category with tensor categories as objects, and the 2-category of bimodule categories for morphisms, see [DSS18].

4 The Cobordism Hypothesis

Given an (∞, n)-category $\mathcal{C}$, let $\mathrm{core}(\mathcal{C})$ denote the (maximal) ∞-groupoid in $\mathcal{C}$ by discarding all non-invertible morphisms.

Theorem 2 (Cobordism Hypothesis)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then, there is an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \simeq \mathrm{core}(\mathcal{C}^{fd})$$

By considering manifolds with n-framings, we obtain an action of the orthogonal group $O(n)$ on the collection of all n-framings. In particular, we get an $O(n)$-action on $\mathrm{Bord}_n^{\mathrm{fr}}$ and hence the (∞, n)-category of framed TFTs. By Theorem 2, this induces an action of the orthogonal group on the underlying ∞-groupoid of fully dualizable objects.

Definition 6. Let X be a topological space and ζ a real vector bundle on X of dimension n. Let M be a smooth manifold with dimension $m \leq n$. An (X, ζ)-structure on M consists of:

1. a continuous map $f : M \to X$ and
2. An isomorphisms of vector bundles $TM \oplus \mathbb{R}^{n-m} \cong f^*\zeta$.

Theorem 3. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals and (X, ζ) where X is a CW complex and ζ a real vector bundle of rank n equipped with an inner product. Let $\tilde{X} \to X$ denote the principal $O(n)$-bundle of orthogonal frames in ζ. Then, there is an equivalence of $(\infty, 0)$-categories*

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{(X,\zeta)}, \mathcal{C}) \simeq \mathrm{Hom}_{O(n)}(\tilde{X}, \mathrm{core}(\mathcal{C}))$$

Some interesting of (X, ζ)-structures are the following.

Definition 7. Let G be a topological group together with continuous homomorphism $\zeta : G \to O(n)$. Let $BG = EG/G$ be the classifying space of G, where EG is the total space of the universal bundle over BG with a free action of G. Consider the associated vector bundle $\zeta_\chi = (\mathbb{R}^n \times EG)/G$ over BG. By a G-structure on M, we then mean an (BG, ζ_χ)-structure.

Example 3.
- Let G be the trivial group. Then, $BG = *$ and ζ is trivial. An G-structure is then a framing. In particular, $\mathrm{Bord}_n^G \simeq \mathrm{Bord}_n^{\mathrm{fr}}$.
- Let $G = O(n)$ with $\chi = \mathrm{id}_{O(n)}$. An $O(n)$-structure is an (unoriented) smooth manifold. In particular, $\mathrm{Bord}_n^{O(n)} \simeq \mathrm{Bord}_n$.
- Let $G = SO(n)$ with $\chi : SO(n) \hookrightarrow O(n)$. An $SO(n)$-structure is an orientation. In particular, $\mathrm{Bord}_n^{SO(n)} \simeq \mathrm{Bord}_n^{or}$.

Corollary. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals, and let $\chi : G \to O(n)$ be a continuous group homomorphism. Then, there is an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \simeq (\mathrm{core}\mathcal{C})^{hG}$$

of G-structured TFTs and homotopy fixed points in $\mathrm{core}(\mathcal{C})$.

Some special choices of G lead to:

- $\mathrm{Bord}_n \simeq \mathrm{Bord}_n^{O(n)}$
- $\mathrm{Bord}_n^{\mathrm{or}} \simeq \mathrm{Bord}_n^{SO(n)}$
- $\mathrm{Bord}_n^{\mathrm{fr}} \simeq \mathrm{Bord}_n^{\{1\}}$

Example 4. In the 1-dimensional case the action of $O(1) = \mathbb{Z}_2$ in $\mathrm{core}\mathcal{C}$ is given by taking the dual, i.e. $X \mapsto X^*$. Then homotopy fixed points correspond to *symmetric* objects. For example, in the category of finite dimensional vector spaces, such objects would be vector spaces V together with a symmetric non-degenerate bilinear form.

A Complete n-fold Segal spaces

Definition 8. A simplicial space $X_\bullet$ is a *Segal space* if the the Segal maps

$$X_n \to X \times^h_{X_0} \cdots \times^h_{X_0} X_1$$

are weak homotopy equivalences.

Let X be a Segal space. Then, we can associate its homotopy 1-category $h_1(X)$ which has the same objects as X and for objects x, y the morphism set $Hom_{h_1(X)}(x,y) := \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\})$. A morphism f in X is called an equivalence if it is invertible in $h_1(X)$. Let X^{equiv} denote the subspace of equivalences in X_1. Then, it is clear that the degenerate map $\delta : X_0 \to X_1$ takes values in X^{equiv}.

Definition 9. A Segal space X is called *complete* if the canonical map $\delta : X_0 \to X^{\mathrm{equiv}}$ is a weak homotopy equivalence.

Complete Segal spaces model $(\infty, 1)$-categories. For (∞, n)-categories, one extendes this notion to that of complete n-fold Segal spaces.

Definition 10. An n-fold simplicial space $X : (\Delta^{\mathrm{op}})^{\times n} \to \mathrm{Sp}$ is called *n-fold Segal space* if for every $i \in \{1, \dots, n\}$

1. for every $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n$ the simplicial space

$$X_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$$

   is a Segal space,

2. for every $k_1, \dots, k_{i-1}$ the $(n-i)$-fold simplicial space

$$X_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet}$$

   is essentially constant[5].

In this setting, one should think of elements in $X_{k_1,\dots,k_n}$ as a colletction of k_i composable morphisms in the ith-direction. The (∞, n)-category which we want to associate to such n-fold Segal spaces should have k-morphisms given by elements in $X_{1,\dots,1,0,\dots,0}$ where $k \leq n$ is the number of 1's.

Definition 11. An n-fold Segal space X is called *complete* if for all $i \in \{1, \dots, n\}$ and every $k_1, \dots, k_{i-1}$ the Segal space

$$X_{k_1,\dots,k_{i-1},\bullet,0,\dots,0}$$

is complete.

Let (C)SSp_n denote the category of (complete) n-fold Segal spaces. Given an n-fold Segal space $X \in \mathrm{SSp}_n$, there exists a (unique up to weak equivalence) complete n-fold Segal space $\widehat{X}$. In fact, this construction $\widehat{(-)} : \mathrm{SSp}_n \to \mathrm{CSSp}_n$ is a left adjoint to the functor $\mathrm{CSSp}_n \hookrightarrow \mathrm{SSp}_n$ which forgets the completeness.

[5] A n-fold simplicial space $X_{\bullet,\dots,\bullet}$ is called *essentially constant* if the degenerate map $X_{0,\dots,0} \to X_{\bullet,\dots,\bullet}$ is a weak equivalence.

B $\mathrm{PreBord}_n$ in [Lur09]

The definition proposed by Lurie is the following:

Definition 12. Let V be a finite dimensional vector space. Elements in $(\mathrm{PreBord}_n^V)_{k_1,\dots,k_n}$ are tuples $(M, (t_0^i \leq \cdots \leq t_{k_i}^i)_{1\leq i\leq n})$ where

- M is a closed n-submanifold in $V \times \mathbb{R}^n$
- The projection map $\pi : M \hookrightarrow V \times \mathbb{R}^n \to \mathbb{R}^n$ is proper.
- For any subset $S \subset \{1, \dots, n\}$ and non-negative integers $j_i \leq k_i$ for each $i \in S$ the projection map onto the S-coordinates $p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ has no critical values on $(t_{j_i}^i)_{i\in S}$.
- At every point $x \in p_{\{i\}}^{-1}(\{t_0^i, \dots, t_{k_i}^i\})$ the map $p_{\{i+1,\dots,n\}}$ is submersive.

However, the resulting $\mathrm{PreBord}_n^V$ is not quite an n-fold Segal space as observed in [CS15]. It satisfies the Segal condition but not the essential constancy (for the relevant case when $n > 1$). To see this, consider Figure 3 as an example of an element in $(\mathrm{PreBord}_2)_{0,1}$. Namely, a torus embedded in $\mathbb{R}^3$ where the red line indicates the cut in the first time coordinate and the green lines the two cut points in the second time coordinate. This element however cannot be connected to an degenerate element, which fails the essentially constancy condition.

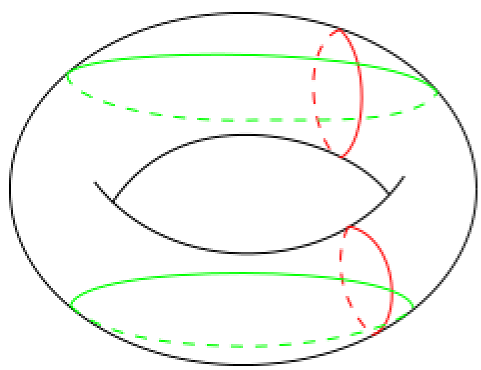


Figure 3

Remark. Suppose we have an element in $(\mathrm{PreBord}_n^V)_{k_1,\dots,k_n}$ as in Definition 3, i.e. a tuple $(M, (t_0^i \leq \cdots \leq t_{k_i}^i)_{1\leq i\leq n})$ such that the map $p_{\{i,\dots,n\}}$ is submersive at every point $x \in p_{\{i\}}^{-1}(\{t_0^i, \dots, t_{k_i}^i\})$. Let now S be any subset in $\{1, \dots, n\}$ and $\{1 \leq j_i \leq k_i\}_{i\in S}$ any collection of integers. Let i_0 be the smallest integer in S. Then, we can write $p_S = pr \circ p_{\{i_0,\dots,i_n\}}$, where pr is the projection onto the S-coordinates. Suppose $x \in M$ such that $p_S(x) = (t_{j_i}^i)_{i\in S}$. In particular, we have $p_{\{i_0\}}(x) = t_{j_{i_0}}^{i_0}$. Thus, $p_{\{i_0,\dots,n\}}$ is submersive at x and therefore p_S is also submersive at x. In particular, p_S has no critical values on $(t_{j_i}^i)_{i\in S}$.

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