Little Cubes and Manifold Topology

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Abstract

In this talk we survey Sections 5.2 and 5.4 of [Lur17]. We introduce two ∞-operads associated to a manifold M: the ∞-operad $\mathbb{E}_M^{\otimes}$ that describes embeddings of disks into M up to continuous deformations, and the ∞-operad $\mathrm{NDisk}(M)^{\otimes}$ of discrete nature that consists only of topologically trivial open subsets of M and their disjoint inclusions into each other. The final section of this talk shows the relation of these operads to each other, and points towards their importance for chiral homology and factorisation algebras on M. In particular, this relation makes precise the slogan that 'locally constant factorisation algebras are equivalent to $\mathbb{E}_k^{\otimes}$-algebras'.

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1 Introduction

1.1 Configurations

Recall that one of our motivating examples at the beginning of this seminar was to formalise 'configurations of points in a space M with summable labels'. Let M be a manifold, let S be a set, and let A be a monoid. A configuration in M of the points of S is a span of maps $A \leftarrow S \xrightarrow{\iota} M$, where ι is an injection. In other words, an A-labelled S-configuration in M is a set of tuples

$$\{(x(s), a(s)) \in M \times A \,|\, s \in S,\ x(s) \neq x(s')\ \forall s \neq s' \in S\}$$

A path of S-configurations in M is a path in the space $C_S(M) = \mathrm{Emb}(S, M)$ – this provides us with an ∞-groupoid $\mathrm{Sing}\,\mathrm{Emb}(S, M)$ of S-configurations in M. If A is a topological monoid, we obtain an ∞-groupoid of A-labelled S-configurations in M as

$$C_S(M; A) = \mathrm{Sing}\left(\mathrm{Emb}(S, M) \times_{\Sigma_{|S|}} A^S\right).$$

However, we would like to assemble all the ∞-groupoids $C_{\langle m \rangle^\circ}(M; A)$ for $m \in \mathbb{N}_0$ into one ∞-category by adding in more morphisms. For example, we would like to allow for any number of points to collide in a single point – this replaces all these points by the point of collision, whose label now becomes the sum of the labels of the incoming points.

This looks very much like the outcome should be an ∞-operad – however, making this idea precise requires significant work. In the special case of $M = \mathbb{R}^k$, this has been achieved originally by Fulton and MacPherson [FM94]. The problem is to find a correct incorporation of enough geometric information about the collisions of points; this leads to the Fulton-MacPherson operads, see [Sal01] for a good treatment in the context of manifolds. The key is to not just remember that a certain number of points collide, but to remember a certain amount of information about how they collide, i.e. relative distances and velocities – points can collide in many different ways.

A different idea to approach the description of collisions arises from the realisation that the main difficulties in turning configurations in M into an operad stem from the singular nature of the points that we map into M. So let us focus on the points for now and forget about labelling them from now on. An idea is to thicken the points a little: instead of embeddings $S \hookrightarrow M$, we consider embeddings $S \times \mathbb{R}^k \hookrightarrow M$, where $k = \dim(M)$. Instead of *the(!) collision* of a certain number of points in M, we can now understand *a choice of(!) collision* of a certain number of thickened points, i.e. disks, in M as a larger disk in M that contains all the original disks. Thus, instead of the spaces $\mathrm{Emb}(\langle m \rangle^\circ, M)$, we ask weather we can turn the spaces $E_{M|m} = \mathrm{Emb}(\langle m \rangle^\circ \times \mathbb{R}^k, M)$ into an ∞-operad $\mathbb{E}_M^\otimes$, where the composition maps are built from embeddings of disks into each other, together with isotopies of embeddings of disks into M.

1.2 Observables in quantum field theory

There is a variant of the spaces $E_{M|k}$ that is important in approaches to *observables* in (topological) quantum field theories [CG17, CG16]: in this approach to field theories on M, one would like to associate to each region U in M the set of observables $A(U)$ that can be measured in U. By invoking the principle of *locality*, it is assumed that observables in any region U should always be constructible from observables in subregions that cover U. Thus, since M is locally euclidean, it would be enough to know observables on open subsets $U \subset M$ where U is a disk in M in the sense that there is some homeomorphism $U \cong \mathbb{R}^k$.

For a generic field theory, it might happen that for an inclusion $U \subset V$ of open subsets of M as above there are strictly more observables supported in V than there are in U. Thus, there is no reason to expect that a continuous change of U (in whatever sense) should lead to equivalent observables. Consequently, in this case we are only interested in the discrete category $\mathrm{Disk}(M)$ of open subsets of M as above with inclusions.

Let $U, V \in \mathrm{Disk}(M)$ be disjoint. We would like to talk about observables on U and V at the same time, so we assume that $\mathcal{C}$ is a monoidal ∞-category. Given an object $W \in \mathrm{Disk}(M)$ with $U, V \subset W$, *a multiplication* of observables in $A(U)$ and $A(V)$ is then given by a morphism

$$A(U) \otimes A(V) \to A(W)\,.$$

Note that there is no prescribed ordering on subsets $U \subset M$, so that the above should be accompanied by a morphism $A(V) \otimes A(U) \to A(W)$, and talking about 'observables on U and observables on V' should be equivalent to talking about 'observables on V and observables on U'. Thus, A should factor through the centre of $\mathcal{C}$, and we may assume right away that $\mathcal{C}$ is symmetric monoidal.

Summarising, Disk(M) should assemble into an ∞-operad, denoted $\mathrm{NDisk}(M)^\otimes$, with morphisms for inclusions with disjoint images, and observables in a quantum field theory should be modelled by algebras over that operad. Finally, one might understand *topological* field theories as those theories whose observables do not (essentially) change by passing to smaller sub-regions in Disk(M), and whose observables assigned to a disk in M do not (essentially) change when the disk is moved around in M continuously. Thus, topological field theories in this sense should be related to both ∞-operads $\mathrm{NDisk}(M)^\otimes$ and $\mathbb{E}_M^\otimes$. This will be made precise in Section 3.2.

Conventions: Embeddings and homeomorphisms of manifolds

We fix some notation:

- Throughout these notes, a *manifold* will mean a compact topological manifold, unless otherwise specified.
- For manifolds M, N, we let $\mathrm{Emb}(M, N)$ denote the topological space of embeddings of M into N – the topology comes from the inclusion $\mathrm{Emb}(M, N) \subset \underline{\mathrm{Mfd}}(M, N)$. Similarly, we let $\mathrm{Homeo}(M, N)$ denote the space of homeomorphisms from M to N.
- We use the shorthand notation

$$\mathrm{Top}(k) \coloneqq \mathrm{Homeo}(\mathbb{R}^k, \mathbb{R}^k)\,.$$

2 Embeddings of disks into M

2.1 Variations on the little cubes operad

Recall that the little cubes operads E_k from last week are built from spaces of *rectilinear* embeddings with disjoint images of k-cubes into a k-cube. As motivated in Section 1.1, we would like to consider embeddings of cubes into a manifold M, where a priori there is no such notion as 'rectilinear'. Thus, we start by dropping this requirement and consider generic embeddings instead. Instead of the standard open k-cube $(-1, 1)^k$, we consider the equivalent space $\mathbb{R}^k$.

Construction 2.1 Let $k \in \mathbb{N}_0$ and define the following topological category denoted ${}^t\mathbb{E}_{\mathrm{BTop}(k)}$:

(1) The objects of ${}^t\mathbb{E}_{\mathrm{BTop}(k)}$ are the finite pointed sets $\langle m \rangle \in \mathcal{F}\mathrm{in}_*$.

(2) We define the mapping spaces in ${}^t\mathbb{E}_{\mathrm{BTop}(k)}$ via

$$\underline{{}^t\mathbb{E}_{\mathrm{BTop}(k)}}(\langle m \rangle, \langle n \rangle) = \coprod_{\alpha\colon \langle m \rangle \to \langle n \rangle} \prod_{1 \leq i \leq n} \mathrm{Emb}\left(\mathbb{R}^k \times \alpha^{-1}(i), \mathbb{R}^k\right).$$

Observe that there is a canonical projection functor ${}^t\mathbb{E}_{\mathrm{BTop}(k)} \to \mathcal{F}\mathrm{in}_*$. ◁

Definition 2.2 *We let* $\mathrm{BTop}(k)^\otimes \coloneqq \mathrm{N}({}^t\mathbb{E}_{\mathrm{BTop}(k)})$ *denote the* ∞*-operad obtained by applying the homotopy-coherent nerve to* ${}^t\mathbb{E}_{\mathrm{BTop}(k)}$.

Remark 2.3 (1) We have

$$\mathrm{BTop}(k)^\otimes_{\langle 0 \rangle} = *\,,$$

where the one object is the unique morphism $\emptyset \to \emptyset$. Even more, this unique object in the fibre $\mathrm{BTop}(k)^\otimes_{\langle 0 \rangle}$ is initial in $\mathrm{BTop}(k)$. In particular, $\mathrm{BTop}(k)^\otimes$ is unital.

(2) We find that the ∞-category underlying $\mathrm{BTop}(k)^{\otimes}$ is the homotopy-coherent nerve

$$\mathrm{BTop}(k)^{\otimes}_{\langle 1\rangle} = \mathrm{N}\big(\,\mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k) \rightrightarrows *\big) =: \mathrm{B}\,\mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k)\,,$$

the homotopy-coherent nerve of the topological category with one object $*$ and whose space of morphisms $* \to *$ is the topological monoid $\mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k)$. Note that we may think of the unique object $*$ as a copy of $\mathbb{R}^k$. ◁

The reason why this ∞-operad is called $\mathrm{BTop}(k)^{\otimes}$ derives from

Theorem 2.4 (Kister-Mazur Theorem, [Lur17, Thm. 5.4.1.5]) *For each $k \in \mathbb{N}_0$, the canonical inclusion* $\mathrm{Top}(k) \hookrightarrow \mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k)$ *is a homotopy equivalence. In fact, the inclusions of topological monoids*

$$\mathrm{O}(k) \hookrightarrow \mathrm{GL}(k) \hookrightarrow \mathrm{Top}(k) \hookrightarrow \mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k)$$

are all homotopy equivalences [AF19, Prop. 2.6].

Corollary 2.5 *The topological monoid* $\mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k)$ *is* grouplike*; that is, the set* $\pi_0\,\mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k)$ *endowed with the induced monoid structure is a group.*

Corollary 2.6 *We infer that* $\mathrm{BTop}(k)^{\otimes}_{\langle 1\rangle}$ *is an ∞-groupoid, i.e. a Kan complex, and moreover, that*

$$\mathrm{BTop}(k)^{\otimes}_{\langle 1\rangle} \simeq \mathrm{B}\big(\,\mathrm{Top}(k)\big)$$

is equivalent to the classifying space of the topological group $\mathrm{Top}(k)$*.*

Example 2.7 The operad $\mathrm{BTop}(1)^{\otimes}$ describes homotopy-associative algebras with an anti-involution; the latter comes from the (essentially unique) orientation-reversing embedding of $\mathbb{R}$ into itself [Lur17, E.g. 5.4.3.5]. ◁

Remark 2.8 Let $\square^k = (-1,1)^k$ denote the topological k-cube. Any choice of homeomorphism $\square^k \to \mathbb{R}^k$ induces an inclusion of ∞-operads

$$\mathbb{E}_k^{\otimes} \to \mathrm{BTop}(k)^{\otimes}\,.$$

However, *this inclusion is not an equivalence*: while in $\mathbb{E}_k^{\otimes}$ the multi-morphisms contain no more information (up to homotopy) than the location of an embedded cube inside another cube – see Lemma 2.2 of Merlin's notes – the morphism spaces in $\mathrm{BTop}(k)$ are non-trivial, even up to homotopy. For instance, an embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^k$ might preserve or reverse orientations on $\mathbb{R}^k$, already leading to multiple path-connected components of mapping spaces. ◁

Construction 2.9 [Lur17, Constr. 2.4.3.1] Let Γ^* be the category of injective morphisms $\langle 1\rangle \to \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$. In other words, an object of Γ^* is a pair $(\langle n\rangle, j)$, where $\langle n\rangle \in \mathcal{F}\mathrm{in}_*$ and $j \in \langle n\rangle^{\circ}$. A morphism $(\langle m\rangle, i) \to (\langle n\rangle, j)$ in Γ^* is a morphism $\alpha\colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$ that maps i to j.

Let $K \in \mathcal{S}\mathrm{et}_{\Delta}$. We define a new simplicial set $K^{\amalg}$ with a morphism $p\colon K^{\amalg} \to \mathrm{N}\mathcal{F}\mathrm{in}_*$ by the following universal property: for every $L \in (\mathcal{S}\mathrm{et}_{\Delta})_{\mathrm{N}\mathcal{F}\mathrm{in}_*}$, there is a canonical bijection

$$\mathrm{Hom}_{\mathrm{N}\mathcal{F}\mathrm{in}_*}(L, K^{\amalg}) \cong \mathcal{S}\mathrm{et}_{\Delta}\,\big(L \underset{\mathrm{N}\mathcal{F}\mathrm{in}_*}{\times} \mathrm{N}\Gamma^*, K\big)\,.$$

Lurie shows in [Lur17, Prop. 2.4.3.3] that if $K = \mathcal{C}$ is an ∞-category, then $p\colon \mathcal{C}^{\amalg} \to \mathrm{N}\mathcal{F}\mathrm{in}_*$ is an ∞-operad. Further, this is even a symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits finite coproducts [Lur17, Rmk. 2.4.3.4].

Unravelling the definition[1], we see that the ∞-operad $\mathcal{C}^{\amalg}$ has as objects finite tuples $(C_1, \ldots, C_m)$ of objects in $\mathcal{C}$, and a morphism $(C_1, \ldots, C_m) \to (C'_1, \ldots, C'_n)$ consists of a map $\alpha\colon \langle m\rangle \to \langle n\rangle$ together with families $\{f_l\colon C_l \to C'_{\alpha(l)}\}_{l\in\alpha^{-1}\langle n\rangle^\circ}$ of morphisms in $\mathcal{C}$. In other words, for every $l' \in \langle n\rangle^\circ$ and every $l \in \langle m\rangle^\circ$ such that $\alpha(l) = l'$ we choose a morphism $C_l \to C'_{l'}$. ◁

Example 2.10 Consider the ∞-groupoid $\mathrm{BTop}(k)$. An object in $\mathrm{BTop}(k)^{\amalg}$ consists of a finite tuple[2] $(\langle m\rangle, \mathbb{R}^k, \ldots, \mathbb{R}^k)$ of copies of the unique object of $\mathrm{BTop}(k)$ (see Remark 2.3). A morphism $(\langle m\rangle, \mathbb{R}^k, \ldots, \mathbb{R}^k) \to (\langle n\rangle, \mathbb{R}^k, \ldots, \mathbb{R}^k)$ consists of a map $\alpha\colon \langle m\rangle \to \langle n\rangle$, and for every $l' \in \langle n\rangle^\circ$ and every $l \in \langle m\rangle^\circ$ such that $\alpha(l) = l'$ an embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^k$. Higher morphisms stem from homotopies of embeddings.

Note that, for example for α the unique active morphism $\langle 2\rangle \to \langle 1\rangle$, we need to choose two embeddings $\mathbb{R}^k \hookrightarrow \mathbb{R}^k$, but in contrast to the morphisms in $\mathrm{BTop}(k)^{\otimes}$ these embeddings are now completely independent of each other; in particular, their images do not need to be disjoint. ◁

We can use the construction $K \mapsto K^{\amalg}$ in order to 'reduce the structure group' of the ∞-operad $\mathrm{BTop}(k)^{\otimes}$. For example, we might want to consider only those embeddings that are homotopic to a special orthogonal transformation, i.e. to an element of the subspace $\mathrm{SO}(k) \hookrightarrow \mathrm{Emb}(\mathbb{R}^k, \mathbb{R}^k)$.

Definition 2.11 [Lur17, Def. 5.4.2.10] *Let B be a Kan complex, and let $\theta\colon B \to \mathrm{BTop}(k)$ be a Kan fibration. We define a new ∞-operad as the fibre product (in $\mathcal{S}\mathrm{et}_\Delta$)*

$$\mathbb{E}_B^{\otimes} \coloneqq \mathrm{BTop}(k)^{\otimes} \underset{\mathrm{BTop}(k)^{\amalg}}{\times} B^{\amalg}\,.$$

Lemma 2.12 *The underlying ∞-category of the ∞-operad $\mathbb{E}_B^{\otimes}$ satisfies*

$$\mathbb{E}^{\otimes}_{B\langle 1\rangle} \cong B\,.$$

Proof. Since limits commute with limits, we have

$$\mathbb{E}^{\otimes}_{B\langle 1\rangle} \cong \mathrm{BTop}(k)^{\otimes}_{\langle 1\rangle} \underset{\mathrm{BTop}(k)^{\amalg}_{\langle 1\rangle}}{\times} B^{\amalg}_{\langle 1\rangle}\,.$$

Also, it holds true that $\mathrm{BTop}(k)^{\otimes}_{\langle 1\rangle} = \mathrm{BTop}(k)^{\amalg}_{\langle 1\rangle} = \mathrm{BTop}(k)$ and $B^{\amalg}_{\langle 1\rangle} = B$. □

Example 2.13 (1) Consider the case of B contractible, e.g. $B = \mathrm{ETop}(k)$ and θ the projection $\mathrm{ETop}(k) \to \mathrm{BTop}(k)$. From this we obtain the ∞-operad

$$\mathbb{E}^{\otimes}_{\mathrm{ETop}(k)} \coloneqq \mathrm{BTop}(k)^{\otimes} \underset{\mathrm{BTop}(k)^{\amalg}}{\times} \mathrm{ETop}(k)^{\amalg}$$

The factor $\mathrm{ETop}(k)^{\amalg}$ makes all information contained in the embeddings $\mathbb{R}^k \hookrightarrow \mathbb{R}^k$ entirely trivial; the only remaining restriction is that embeddings into the same copy of $\mathbb{R}^k$ must have disjoint images. For example, we can contract all embeddings onto the image of the origin. Thus, at least heuristically, all information left in the morphism spaces in $\mathbb{E}_{\mathrm{ETop}(k)}$ is given by specifying configurations in $\mathbb{R}^k$. In the last talk, we have seen that the configuration spaces $C_{\langle m\rangle^\circ}(\mathbb{R}^k)$ are equivalent to the level spaces of the little k-cubes operad E_k. Via the usual passage from topological

[1]Consider $L = (\Delta^0 \xrightarrow{\langle m\rangle} \mathrm{N}\mathcal{F}\mathrm{in}_*)$ in order to obtain the objects over $\langle m\rangle$, consider $L = (\Delta^1 \xrightarrow{\alpha} \mathrm{N}\mathcal{F}\mathrm{in}_*)$ in order to obtain the morphisms over α, etc.

[2]We have added $\langle m\rangle$ to the notation here for the reader's convenience.

operads to ∞-operads, the latter gives rise to an ∞-operad[3] $\mathbb{E}_k^{\otimes}$. There is an equivalence of ∞-operads [Lur17, Rmk. 5.4.2.15]

$$\mathbb{E}^{\otimes}_{\mathrm{ETop}(k)} \simeq \mathbb{E}_k^{\otimes} \,.$$

(2) [Lur17, Rmk. 5.4.2.16] We consider the inclusion $\mathrm{SO}(k) \hookrightarrow \mathrm{Top}(k)$. It gives rise to a Kan fibration $\theta\colon \mathrm{BSO}(k) \to \mathrm{BTop}(k)$ and thus to an ∞-operad $\mathbb{E}^{\otimes}_{\mathrm{BSO}(k)}$. Similarly to the preceding example, the ∞-operad $\mathbb{E}^{\otimes}_{\mathrm{BSO}(k)}$ is equivalent to a more hands-on ∞-operad: Let $B(1)$ denote the unit open ball in $\mathbb{R}^k$. For a finite set S, define the topological subspace

$$\mathrm{Isom}^+\big(B(1)\times S, B(1)\big) \subset \mathrm{Emb}\big(B(1)\times S, B(1)\big)$$

of those embeddings ι whose restriction to each component $B(1)\times\{s\}$, $s\in S$, is an affine transformation of the form

$$\iota_{B(1)\times\{s\}} = \lambda_s \cdot \gamma_s + v_s \qquad \lambda_s \in \mathbb{R}_{>0},\ \gamma_s \in \mathrm{SO}(k),\ v_s \in \mathbb{R}^k \quad \forall\, s\in S$$

These mapping spaces assemble into a topological operad, whose associated ∞-operad $\mathbb{E}^{\otimes}_{k,fr}$ is called the *∞-operad of framed k-disks*. There is an equivalence of ∞-operads

$$\mathbb{E}^{\otimes}_{\mathrm{BSO}(k)} \simeq \mathbb{E}^{\otimes}_{k,\mathrm{fr}} \,.$$

The framed disk operads are important: for example, for $k=2$ they describe BV-algebras.

(3) Given two Kan complexes B, B' and Kan fibrations $\theta\colon B \to \mathrm{BTop}(k)$ and $\theta'\colon B' \to \mathrm{BTop}(k')$, there is an equivalence of ∞-operads [Lur17, Rmk. 5.4.2.14]

$$\mathbb{E}^{\otimes}_{B\times B'} \simeq \mathbb{E}^{\otimes}_{B} \otimes \mathbb{E}^{\otimes}_{B'} \,,$$

which, for the special case of B, B' contractible, implies that there is an equivalence

$$\mathbb{E}^{\otimes}_{k+k'} \simeq \mathbb{E}^{\otimes}_{k} \otimes \mathbb{E}^{\otimes}_{k'} \,.$$

This is often called the *Dunn Additivity Theorem* [Lur17, Rmk. 5.4.2.14]. ◁

2.2 The ∞-operad $\mathbb{E}_M$

We now come to the definition of the operad $\mathbb{E}^{\otimes}_M$ associated to a topological manifold M. Given M, we define a topological category $\mathcal{C}_M$ with two objects $\mathbb{R}^k$, M, and the following topological morphism spaces:

$$\begin{aligned} \underline{\mathcal{C}_M}(\mathbb{R}^k,\mathbb{R}^k) &= \mathrm{Emb}(\mathbb{R}^k,\mathbb{R}^k)\,, & \underline{\mathcal{C}_M}(\mathbb{R}^k, M) &= \mathrm{Emb}(\mathbb{R}^k, M)\,,\\ \underline{\mathcal{C}_M}(M,M) &= \{1_M\} & \underline{\mathcal{C}_M}(M,\mathbb{R}^k) &= \emptyset\,. \end{aligned}$$

Observe that there is a canonical inclusion $\mathrm{BTop}(k) \hookrightarrow \mathrm{N}\mathcal{C}_M$. Hence we can define the simplicial set

$$B_M \coloneqq \mathrm{BTop}(k) \underset{\mathrm{N}\mathcal{C}_M}{\times} (\mathrm{N}\mathcal{C}_M)_{/M}\,.$$

Lemma 2.14 *The simplicial set B_M is a Kan complex, and the projection $\theta\colon B_M \to \mathrm{BTop}(k)$ is a Kan fibration.*

[3] See Section 3 of Merlin's notes and [Lur17, Sect. 5.1] for more on the ∞-operads $\mathbb{E}_k^{\otimes}$.

Proof. Since in the pullback diagram

$$\begin{array}{ccc} B_M & \longrightarrow & (\mathrm{N}\mathcal{C}_M)_{/M} \\ {\scriptstyle (\mathrm{RFib})}\big\downarrow & & p\big\downarrow{\scriptstyle \mathrm{RFib}} \\ \mathrm{BTop}(k) & \longrightarrow & \mathrm{N}\mathcal{C}_M \end{array} \tag{2.15}$$

the right vertical morphism is a right fibration (dual of [Lur09, Prop. 2.1.2.2]), so is the left vertical morphism. It is then also a Kan fibration by Theorem 2.4 and [Lur09, Lem. 2.1.3.3] (left/right fibrations whose target is a Kan complex are Kan fibrations). □

Remark 2.16 The right vertical map in Diagram (2.15) is in particular an inner fibration, and all vertices of the diagram are ∞-categories. Hence, this pullback is a model for the homotopy pullback of the diagram in the Joyal model structure on $\mathcal{S}\mathrm{et}_\Delta$. ◁

Morally, an object in B_M is an embedding $\iota\colon \mathbb{R}^k \hookrightarrow M$, and a morphism $\iota \to \iota'$ is an embedding $\jmath\colon \mathbb{R}^k \hookrightarrow \mathbb{R}^k$ with a homotopy $\iota \to \iota' \circ \jmath$.

Remark 2.17 Note that it might be tempting to use $\widetilde{B}_M \coloneqq \mathrm{Sing}(\mathrm{Emb}(\mathbb{R}^k, M))$ instead of B_M. This is also a Kan complex describing embeddings of $\mathbb{R}^k$ into M, but it does not come with a Kan fibration to $\mathrm{BTop}(k)$, which we need in order to apply the construction behind Definition 2.11. ◁

Definition 2.18 *Given a topological manifold M of dimension $k \in \mathbb{N}_0$, we define an ∞-operad*

$$\mathbb{E}_M^\otimes \coloneqq \mathbb{E}_{B_M}^\otimes = \mathrm{BTop}(k)^\otimes \underset{\mathrm{BTop}(k)^{\amalg}}{\times} B_M^{\amalg}\,.$$

Let us have a closer look at the ∞-category $\mathbb{E}_M^\otimes$. Its objects over $\langle m\rangle \in \mathcal{F}\mathrm{in}_*$ consist of m copies of the unique object $\mathbb{R}^k$ of $\mathrm{BTop}(k)^\otimes$ and an m-tuple $(\iota_i\colon \mathbb{R}^k \hookrightarrow M)_{i=1,\dots m}$ of objects in B_M; we thus only need to remember the m-tuple ι_i of embeddings of $\mathbb{R}^k$ into M. Observe, however, that the embeddings ι_i do *not* necessarily have disjoint images.

A morphism $(\iota_i)_{i\in\langle m\rangle^\circ} \to \iota$ in $\mathbb{E}_M^\otimes$ that covers $\alpha\colon \langle m\rangle \to \langle 1\rangle$ consists of a collection of embeddings $(\jmath_i\colon \mathbb{R}^k \hookrightarrow \mathbb{R}^k)_{i\in\langle m\rangle^\circ}$ *with disjoint images* and homotopies $h_i\colon \iota_i \to \iota\circ\jmath_i$ of embeddings $\mathbb{R}^k \hookrightarrow M$. In particular, observe that since $\iota\colon \mathbb{R}^k \hookrightarrow M$ is an embedding and since $(\jmath)_i$ have disjoint images, the family of embeddings $(\iota\circ\jmath_i\colon \mathbb{R}^k \hookrightarrow M)_{i\in\langle m\rangle^\circ}$ always has disjoint images in M. That is, in order to compose a family of embedded disks in M (with possibly intersecting images), we first have to specify a way of pulling the disks apart from each other, and then we have to specify an embedded disk in M that contains all the new embedded disks.

Proposition 2.19 [Lur17, Rmk. 5.4.5.2] *There is an isomorphism $B_M \cong \mathrm{Sing}(M)$ in $\mathcal{H}$, the homotopy category of the ∞-category of spaces.*

Corollary 2.20 *For any $k \in \mathbb{N}_0$, there is an equivalence of ∞-operads*

$$\mathbb{E}_{\mathbb{R}^k}^\otimes \simeq \mathbb{E}_k^\otimes\,,$$

which follows from $B_{\mathbb{R}^k} \simeq \Delta^0 \simeq \mathrm{ETop}(k)$, together with Example 2.11.1.

Example 2.21 In particular, we have $\mathbb{E}_1 \simeq \mathbb{E}_{\mathbb{R}}^\otimes \not\simeq \mathbb{E}_{\mathrm{BTop}(1)}^\otimes$: the ∞-operad on the left describes homotopy-associative algebras, while the ∞-operad on the right describes homotopy-associative algebras with an anti-involution. Of course the ∞-operad $\mathbb{E}_{\mathbb{R}}^\otimes$ still contains orientation-reversing embeddings $\mathbb{R} \hookrightarrow \mathbb{R}$ as objects, but our choice of $\theta\colon B_{\mathbb{R}} \simeq \mathrm{ETop}(1) \to \mathrm{BTop}(1)$ adds in a contractible space of paths that connect this object to the identity embedding $1_{\mathbb{R}}$. ◁

This completes the construction of the operads motivated in Section 1.1. Compared to configurations of points in M, the operads $\mathbb{E}_M^{\otimes}$ overcount: they contain not just information about the location of a thickened point in M, but also the whole embeddings behind. we can trivialise parts of this additional information, if desired, by certain choices of $\theta\colon B \to \mathrm{BTop}(k)$.

3 Disks in M

We now pursue the ideas outlined in Section 1.2 and construct an operad $\mathrm{NDisk}(M)^{\otimes}$ of open subsets of a manifold M that are homeomorphic to $\mathbb{R}^k$. The two essential differences to the construction of the operad $\mathbb{E}_M$ are that we do not care about *how* that disk gets into the manifold M (i.e. we do not specify any actual embeddings), and that we do no longer care about continuous deformations of the disks inside M.

The operad $\mathrm{NDisk}(M)^{\otimes}$ is strongly related to factorisation algebras – see, for instance, [AF19, CG17, CG16, Gin15] and the next talk in this seminar. Finally, we will give a precise statement about the relation of the two operads $\mathbb{E}_M^{\otimes}$ and $\mathrm{NDisk}(M)^{\otimes}$ that we can associate to a manifold M.

3.1 The ∞-operad $\mathrm{NDisk}(M)^{\otimes}$

Motivated by our discussion in Section 1.2, we make the following definition.

Definition 3.1 [Lur17, Def. 5.4.5.6] *Let M be a manifold of dimension $k \in \mathbb{N}_0$.*

(1) Let $\mathrm{Disk}(M)$ denote the partially ordered set of those open subsets of M that are homeomorphic to $\mathbb{R}^k$. The partial order is given by inclusion of sets.

(2) Let $\mathrm{NDisk}(M)^{\otimes} \subset \mathrm{NDisk}(M)^{\amalg}$ denote the subcategory spanned by those morphisms $(U_1, \ldots, U_m) \to (V_1, \ldots V_n)$ with the property that for every pair $i, j \in \langle m \rangle^\circ$ that have the same image in $\langle n \rangle^\circ$ the subsets U_i and U_j are disjoint.

Remark 3.2 The ∞-category $\mathrm{NDisk}(M)^{\otimes}$ is precisely the ∞-operad associated[4] to the (ordinary) coloured operad $\mathcal{O}_M$ whose colours are the objects $U \in \mathrm{NDisk}(M)^{\otimes}$, and whose sets of multi-morphisms read as

$$\mathrm{Mul}_{\mathcal{O}_M}\big((U_1, \ldots, U_m), V\big) = \begin{cases} *\,, & (U_1, \ldots, U_m) \text{ are pairwise disjoint and } U_i \subset V \;\forall\, i \in \langle m \rangle^\circ\,, \\ \emptyset\,, & \text{otherwise}\,. \end{cases}$$

Our goal for the remainder of this talk is now to compare the ∞-operads $\mathbb{E}_M^{\otimes}$ and $\mathrm{NDisk}(M)^{\otimes}$. In order to achieve this, we first define an intermediate operad $\mathrm{NDisk}'(M)^{\otimes}$: let $\mathrm{Disk}'(M)$ denote the category whose objects are embeddings $\mathbb{R}^k \hookrightarrow M$ and whose morphisms are (strictly) commutative triangles of embeddings

$$\begin{array}{ccc} \mathbb{R}^k & \overset{f}{\hookrightarrow} & \mathbb{R}^k \\ & \searrow \quad \swarrow & \\ & M & \end{array} \tag{3.3}$$

This comes with a functor

$$\pi\colon\ \mathrm{Disk}'(M) \to \mathrm{Disk}(M)$$

[4] This construction was presented in Hendrik's talk.

that sends an embedding $\jmath\colon \mathbb{R}^k \hookrightarrow M$ to its image $\jmath(\mathbb{R}^k)$. Since all morphisms in Diagram (3.3) are embeddings, the functor π is an equivalence of categories.

The category $\mathrm{Disk}'(M)$ gives rise to a coloured operad $\mathcal{O}'_M$ in a fashion analogous to Remark 3.2: its colours are the objects $\jmath\colon \mathbb{R}^k \hookrightarrow M$ of $\mathrm{Disk}'(M)$, and its multi-morphisms read as

$$\mathrm{Mul}_{\mathcal{O}'_M}\big((\jmath_1,\dots,\jmath_m),\jmath\big) = \begin{cases} *\,, & \jmath_i(\mathbb{R}^k)\cap\jmath_j(\mathbb{R}^k) = \emptyset\ \forall\, i\neq j \text{ and } \jmath_i(\mathbb{R}^k)\subset\jmath(\mathbb{R}^k)\ \forall i\in\langle m\rangle^\circ\,,\\ \emptyset\,, & \text{otherwise}\,. \end{cases}$$

Let $\mathrm{NDisk}'(M)^\otimes$ be the ∞-operad associated to the operad $\mathcal{O}'_M$. By construction, he functor π induces an equivalence

$$\pi\colon\ \mathrm{NDisk}'(M)^\otimes \overset{\sim}{\longrightarrow} \mathrm{NDisk}(M)^\otimes$$

of ∞-operads, and we can choose a homotopy-inverse ψ for π.

On the other hand, there is a canonical inclusion

$$\phi\colon\ \mathrm{NDisk}'(M)^\otimes \to \mathbb{E}_M^\otimes\,,$$

and composing this with ψ yields a morphism (canonical up to the contractible choice of ψ) of ∞-operads

$$\Psi\colon\ \mathrm{NDisk}(M)^\otimes \to \mathbb{E}_M^\otimes\,.$$

This morphism is not an equivalence: for instance, the underlying ∞-category of $\mathrm{NDisk}(M)^\otimes$ is $\mathrm{NDisk}(M)$, which is not a Kan complex, but the underlying ∞-category of $\mathbb{E}_M^\otimes$ is $B_M \simeq \mathrm{Sing}(M)$, which is a Kan complex; these cannot be equivalent as ∞-categories.

Nevertheless, the morphism Ψ establishes a very important relation between $\mathrm{NDisk}(M)^\otimes$ and $\mathbb{E}_M^\otimes$, which we state and (partially) prove in the next subsection.

3.2 Algebras over $\mathbb{E}_M^\otimes$ and $\mathrm{NDisk}(M)^\otimes$

The crucial difference between $\mathrm{NDisk}(M)^\otimes$ and $\mathbb{E}_M^\otimes$, or at least of their underlying ∞-categories, is that in $\mathbb{E}_M^\otimes$ there exists a homotopy inverse to every embedding of a disk in M into another disk in M – this is once again the Kister-Mazur Theorem 2.4. It turns out that $\mathbb{E}_M^\otimes$ is obtained from $\mathrm{NDisk}(M)^\otimes$ by adding an inverse to each of the above morphisms; this is made precise by the following theorem:

Theorem 3.4 [Lur17, Thm. 5.4.5.9] *Let M be a manifold, and let $\mathcal{C}^\otimes$ be an ∞-operad. The morphism*

$$\Psi\colon\ \mathrm{NDisk}(M)^\otimes \to \mathbb{E}_M^\otimes$$

induces a fully faithful embedding

$$\Psi^*\colon\ \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathrm{NDisk}(M)}(\mathcal{C})\,.$$

Its essential image is the subcategory spanned by the locally constant $\mathrm{NDisk}(M)^\otimes$*-algebras in* $\mathcal{C}^\otimes$*.*

Remark 3.5 The specification *locally constant* of an algebra has a precise meaning, which is given in [Lur17, Def. 2.3.3.20]. Stating this precisely requires the technical notion of a *weak approximation of ∞-operads* [Lur17, Def. 2.3.3.6], which we will not introduce here.

However, for an ∞-operad $p\colon \mathcal{O}^\otimes \to \mathrm{N}\mathcal{F}\mathrm{in}_*$ the fact that an $\mathcal{O}$-algebra $A\colon \mathcal{O}^\otimes \to \mathcal{C}^\otimes$ is locally constant implies in particular that for every morphism f in $\mathcal{O}^\otimes$ with $p(f) = 1_{\langle 1\rangle}$ the image $A(f)$ is an equivalence in $\mathcal{C}^\otimes$.

In our case, this means that the value of A on a disk in M only changes up to equivalence under continuous changes of the disk. $\triangleleft$

The proof of Theorem 3.4 rests on invoking [Lur17, Prop. 2.3.4.5], a statement about weak approximations of ∞-operads. In order to apply it to the present situation, one first needs to show that Ψ is a weak approximation of ∞-operads, which is done in [Lur17, Lem. 5.4.5.11]. Second, an lastly, one needs to prove that Ψ induces an equivalence of the underlying ∞-categories:

Lemma 3.6 *For any manifold M, the morphism $\psi := \Psi_{\langle 1 \rangle} \colon \mathrm{NDisk}(M) \to B_M \simeq \mathrm{Sing}(M)$ is a weak homotopy equivalence.*

Note that here we talk about weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$ with the Quillen model structure for ∞-*groupoids*, so that this is equivalent to the claim that B_M is equivalent as an ∞-category to the groupoid completion of $\mathrm{NDisk}(M)$, as we claimed in the introduction to this subsection. The proof of Lemma 3.6 relies on the ∞-categorical Seifert-van Kampen Theorem 3.7:

Theorem 3.7 [Lur17, Thm. A.3.1] *Let X be a topological space, and let $\mathcal{U}(X)$ denote the partially ordered set of open subsets of X. Let $\mathcal{C}$ be a small category, and let $F \colon \mathcal{C} \to \mathcal{U}(X)$ be a functor. For each $x \in X$, let $\mathcal{C}_x \subset \mathcal{C}$ be the full subcategory spanned by those objects $c \in \mathcal{C}$ with $x \in F(c)$. Assume that for every $x \in X$ the simplicial set $\mathrm{N}\mathcal{C}_x$ is weakly contractible (weakly equivalent to Δ^0 in the Quillen model structure on $\mathcal{S}\mathrm{et}_\Delta$). Observe that for every $c \in \mathcal{C}$ there is a canonical morphism $F(c) \hookrightarrow X$ of topological spaces. In this situation, the resulting morphism*

$$\mathrm{hocolim}(\mathrm{Sing} \circ F \colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta) \longrightarrow \mathrm{Sing}(X)$$

is a weak homotopy equivalence in the Quillen model structure on $\mathcal{S}\mathrm{et}_\Delta$.

With this at hand, we can prove Lemma 3.6:

Proof of Lemma 3.6. The construction $U \mapsto B_U$ yields a functor $F_B \colon \mathrm{Disk}(M) \to \mathcal{S}\mathrm{et}_\Delta$. Via the ∞-categorical Grothendieck construction, this gives rise to a cocartesian fibration $q \colon X \to \mathrm{NDisk}(M)$ (this should even be a left fibration since F_B is actually valued in spaces). The fibre of q over an object $U \in \mathrm{NDisk}(M)$ is the Kan complex B_U, which is contractible as a consequence of Proposition 2.19. Then, by [Lur09, 2.1.3.4], q is a trivial Kan fibration.

The inclusions $U \subset M$ induces a morphism $\phi \colon X \to B_M$. Further, observe that q admits a section $s \colon \mathrm{NDisk}(M) \to X$: this is essentially given by choosing an inverse equivalence for the functor $\pi \colon \mathrm{Disk}'(M) \to \mathrm{Disk}(M)$, i.e. by choosing a homeomorphism $\mathbb{R}^k \cong U$ for each object $U \in \mathrm{NDisk}(M)$. By construction, $\psi = \phi \circ s$, and since $s \colon \mathrm{NDisk}(M) \to X$ is a weak equivalence, it now suffices to show that $\phi \colon X \to B_M$ is a weak equivalence. Note that X (or rather a Kan fibrant replacement thereof) is a model for the (∞-categorical) colimit of F_B [Lur09, Cor. 3.3.4.6]. Consequently, if we can show that B_M, together with the inclusions $B_U \hookrightarrow B_M$ is a colimit of F_B in the ∞-category $\mathcal{S}$ of spaces, then it will follow that ϕ is a weak equivalence as claimed, by the essential uniqueness of colimits.

Using again Proposition 2.19 to replace the spaces B_U and B_M by $\mathrm{Sing}(U)$ and $\mathrm{Sing}(M)$, we need to show that

$$\mathrm{hocolim}\big(\mathrm{Sing} \colon \mathrm{NDisk}(M) \to \mathcal{S}\mathrm{et}_\Delta\big) \simeq \mathrm{Sing}(M)\,.$$

This will follow immediately from Theorem 3.7, but in order to apply it, we need to check that for every $x \in M$, the partially ordered set

$$\mathrm{Sing}_x := \big\{U \in \mathrm{NDisk}(M) \,\big|\, x \in U\big\}$$

is weakly contractible. Since M is locally euclidean, it is cofiltered (each finite intersection of open disks around x contains a smaller open disk), and hence it is weakly contractible. □

Remark 3.8 (1) Locally constant $\mathrm{NDisk}(M)^{\otimes}$-algebras in a symmetric monoidal ∞-category are also called *locally constant factorisation algebras on M*. Factorisation algebras will be introduced as certain cosheaves in the following talk. (Note that local constancy implies the cosheaf property.)

(2) Factorisation algebras play an important role in topology [AF19, Gin15] and quantum field theory [CG17, CG16].

(3) For $M = \mathbb{R}^k$, Theorem 3.6 is often phrased as the slogan that *locally constant factorisation algebras on $\mathbb{R}^k$ are the same as $\mathbb{E}_k$-algebras.*

Remark 3.9 Given $\theta\colon B \to \mathrm{BTop}(k)$ and a map $\tau\colon M \to B$, we obtain a morphism $\tau^{\otimes}\colon \mathbb{E}_M^{\otimes} \to \mathbb{E}_B^{\otimes}$. For every ∞-operad, this induces a map

$$\Psi^* \circ \tau^*\colon\ \mathrm{Alg}_{\mathbb{E}_B}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{NDisk}(M)}(\mathcal{C})\,.$$

For example, if M is smooth, the tangential bundle TM always yields a classifying map $TM\colon M \to \mathrm{BO}(k)$. An *orientation* on M is equivalent to a lift of this map to $\mathrm{BSO}(k)$, giving a map

$$\mathrm{Alg}_{\mathbb{E}_{k,\mathrm{fr}}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{NDisk}(M)}(\mathcal{C})\,.$$

A *framing* on M is equivalent to a lift of TM to $\mathrm{ETop}(k)$, or equivalently to giving a trivialisation of the tangent bundle, giving a map

$$\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_M}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{NDisk}(M)}(\mathcal{C})$$

from $\mathbb{E}_k$-algebras to locally constant factorisation algebras on M. This will be interesting in subsequent talks. ◁

A The definition of an ∞-operad

For convenience and for looking up, we copy Lurie's definition of an ∞-operad:

Definition A.1 [Lur17, Def. 2.1.1.10] *An* ∞-operad *is a functor* $p\colon \mathcal{O}^{\otimes} \to \mathrm{N}\mathcal{F}\mathrm{in}$ *between ∞-categories satisfying the following conditions:*

(1) For every inert morphism $f\colon \langle m\rangle \to \langle n\rangle$ and every object $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, there exists a p-cocartesian lift $\overline{f}\colon C \to C'$ in $\mathcal{O}^{\otimes}$ of f to $\mathcal{O}^{\otimes}$. We denote the functor between fibres induced by this lift by $f_!\colon \mathcal{O}^{\otimes}_{\langle m\rangle} \to \mathcal{O}^{\otimes}_{\langle n\rangle}$.

(2) For $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and $f\colon \langle m\rangle \to \langle n\rangle$, let $\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C')$ denote the union of those path-connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ that lie over f. Any choice of p-cocartesian lifts $\overline{\rho^i}\colon C' \to C'_i$ of the morphisms $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ for $i = 1, \ldots, n$ induces a homotopy equivalence

$$\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C') \longrightarrow \prod_{i=1,\ldots,n} \mathrm{Map}^{\rho^i\circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)\,.$$

(3) For every finite collection of objects $C_1, \ldots, C_n \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$, there exists an object $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and a collection of p-cocartesian morphisms $C \to C_i$ that cover $\rho^i\colon \langle n\rangle \to \langle 1\rangle$.

The fibre $\mathcal{O}^{\otimes}_{\langle 1\rangle} =: \mathcal{O}$ is the underlying ∞-category *of the ∞-operad $p\colon \mathcal{O}^{\otimes} \to \mathrm{N}\mathcal{F}\mathrm{in}$.*

Recall also that there is a functorial nerve-type construction that allows us to produce ∞-operads in the above sense from any coloured topological or simplicial operad.

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