Last time:

- $R = F\langle Y, Y^{-1}\rangle$ PV ring for $A \in F^{n\times n}$
- $F \subseteq S$ ∂-ring extension with
  1. S integral domain + $K_{\mathrm{Quot}(S)} = K_F$
  2. $\exists\ X \in GL(n,S) : \partial(X) = A\cdot X$

$\Rightarrow$ $U := (S \otimes_F R)^\partial$ f.g. reduced K-algebra with

$$\Theta : S \otimes_K U \longrightarrow S \otimes_F R,\ s\otimes u \mapsto (s\otimes 1)\cdot u$$

isomorphism of S-algebras.

Consequences:

- R, R' PV rings for $A \in F^{n \times n}$, then
$$\mathrm{Hom}_K(U, -) \cong \mathrm{Isom}^{\partial}_{F\otimes_K -}(R \otimes_K -, R' \otimes_K -)$$
functor of points of the unique affine K-variety X with $\mathcal{O}(X) \cong U$.

- In particular, for $R = R'$, $U = (R \otimes_F R)^{\partial}$
$$\mathrm{Hom}_K(U, -) \cong \mathrm{Aut}^{\partial}_{F\otimes_K -}(R \otimes_K -)$$
functorial group str. (*)

functor of points of K-variety G with $\mathcal{O}(G) \cong U$.

HW6.1+(*) $\Rightarrow$ G is a group object in Var_K, i.e., a linear algebraic group $/K$ with
$$G(K) \cong \mathrm{Gal}^{\partial}(R/F)$$

- More precisely: $U = K\langle Z, Z^{-1}\rangle$ with $Z := Y^{-1} \otimes Y \in GL(n, U)$. Given $\sigma \in G(K) = \mathrm{Hom}_K(U, K)$, the corresponding ∂-automorphism of R is obtained via:

$$
\begin{array}{ccccccc}
R & \longrightarrow & R \otimes_F R & \xrightarrow{\Theta^{-1}} & R \otimes_K U & \xrightarrow{1 \otimes \sigma} & R \qquad (+) \\
r & \longmapsto & 1 \otimes r & & & & \\
Y & \longmapsto & 1 \otimes Y & \longmapsto & Y \otimes Z & \longmapsto & Y \cdot \sigma(Z) \\
& & \overset{\shortparallel}{(Y \otimes 1) \cdot Z} & & & &
\end{array}
$$

<u>So</u>: σ acts on $Y \in GL(n, R)$ via right multiplication by the matrix $\sigma(Z) \in GL(n, K)$.

<u>So far</u>: Geometric interpretation of $U = (R \otimes_F R)^\partial$ as coordinate ring of ∂-Galois group.

We will now also provide a geometric interpretation of the isomorphism

$$\Theta : R \otimes_K U \xrightarrow{\cong} R \otimes_F R$$

Definition 5.7. Let G be a group and let M a set equipped with a G-action

$$G \times M \longrightarrow M, \quad (\sigma, m) \longmapsto \sigma.m \quad .$$

Then M is called a <u>G-torsor</u> if the action is simply transitive: for every $m, m' \in M$ there exists a unique $\sigma \in G$: $\sigma.m = m'$. Equivalently, M is a G-torsor, if the map

$$G \times M \longrightarrow M \times M, \quad (\sigma, m) \longmapsto (\sigma.m, m)$$

is a bijection.

Example 5.8. Let F field and E/F finite Galois extension with Galois group G. Let $x \in E$ primitive element of E/F with minimal polynomial $f(T) \in F[T]$ such that $E = F[T]/(f(T))$. Then

$$f(T) = (T - \lambda_1) \cdots (T - \lambda_n) \qquad \text{in } E[T]$$

and G acts simply transitively on the set $M = \{\lambda_1, \dots, \lambda_n\}$ of roots of $f(T)$ in E making M a G-torsor.

This isomorphism

$$\Theta : R \otimes_K U \xrightarrow{\cong} R \otimes_F R$$

amounts to an analog of Example 5.8 in ∂-Galois theory:

Since R is finitely generated, reduced F-algebra, we would like to interpret it as the coordinate ring of an affine "F-variety". <u>Issue</u>: F is not algebraically closed. To resolve this issue, we proceed as indicated in Remark 4.8 and define the category

$$\mathrm{Aff}_F = \left\{ \begin{matrix} \text{affine} \\ F\text{-schemes} \end{matrix} \right\} \overset{(*)}{:=} (\mathrm{Calg}_F)^{op}$$

of affine F-schemes. Denote by X the affine F-scheme which, via $(*)$, corresponds to the F-algebra R. Further, let $G_F \in \mathrm{Aff}_F$ denote the group object that corresponds to $F \otimes_K U$ (an affine F-group scheme).

We then restrict the inverse $\Psi : R \otimes_F R \to R \otimes_K U$ along $R \hookrightarrow R \otimes_F R$, $r \mapsto 1 \otimes r$, to obtain a morphism

$$R \longrightarrow R \otimes_k U \; (\cong R \otimes_F (F \otimes_k U))$$

in Calg_F via $(*)$ becomes a morphism

$$\gamma : X \times G_F \longrightarrow X$$

in Aff_F. It follows from $(+)$ that this morphism defines a (right-) action of G_F on X in Aff_F.

The isomorphism Ψ then corresponds via $(*)$ to an isomorphism

$$X \times G_F \xrightarrow[\cong]{(\pi, \gamma)} X \times X$$

where $\pi : X \times G_F \to X$ projection onto X.

We arrived at:

Theorem 5.9. The affine F-scheme X corresponding via $(*)$ to a Picard-Vessiot ring R/F is a torsor in Aff_F for the action of the ∂-Galois group scheme G_F on X given by γ.

This theorem, also called the "Torsor Theorem", captures the algebraic geometric content of a Picard-Vessiot ring.

Cor 5.10. Let X be the affine F-scheme, corresponding via (*) to a Picard-Vessiot ring R/F. Suppose that L/F is an F-algebra such $X(L) \neq \emptyset$. Then there is an isomorphism of affine L-schemes

$$X_L \cong G_L$$

$$L \otimes_F R \qquad\qquad L \otimes_u u$$

We have further that $X(\bar{F}) \neq \emptyset$ so that

$$X_{\bar{F}} \cong G_{\bar{F}}$$

isomorphism of affine $\bar{F}$-varieties.

Proof HW

Caution While, for an F-algebra L, the ~~set~~ $X(L)$ is always a $G_F(L)$-torsor, we may have $X(L) = \emptyset$. In this case, it is not true that $X_L \cong G_L$!

Example 5.11. Consider $F = K(t)$ and the ∂-equation

$$\partial(y) = \frac{1}{nt} y \quad \text{for } n \in \mathbb{N},\ n \geq 2.$$

Then, we have seen that

$$R = F[Y]/(Y^n - t) = E$$

is a PV ring. The ∂-Galois group is given by $G = \mu_n(K)$ with coordinate ring

$$U = K[Z]/(Z^n - 1).$$

We have

$$X(F) = \mathrm{Hom}_F(R, F) = \emptyset$$

since the equation $Y^n - t = 0$ does not have a solution in F. In $\overline{F}$, the equation has a solution denoted by $\sqrt[n]{t}$. We have an isomorphism

$$\begin{array}{ccc} \overline{F}[Y]/(Y^n - t) & \xrightarrow{\cong} & \overline{F}[Z]/(Z^n - 1) \\ Y & \longmapsto & \sqrt[n]{t} \cdot Z \end{array}$$

corresponding to $X_{\overline{F}} \cong G_{\overline{F}}$ of Cor 5.10.

§6. The ∂-Galois correspondence

In ordinary Galois theory, for any given finite Galois extension E/F, we have

$$\left\{\begin{matrix} F \subseteq M \subseteq E \\ \text{intermediate} \\ \text{fields} \end{matrix}\right\} \cong \left\{\begin{matrix} H \leq \mathrm{Gal}(E/F) \\ \text{subgroups} \end{matrix}\right\}^{op}$$

We establish the analog in ∂-Galois theory:

> **Theorem 6.1.** Let E/F Picard-Vessiot extension.
> Then there is bijective correspondence
>
> $$\left\{\begin{matrix} F \subseteq M \subseteq E \\ \text{intermediate} \\ \partial\text{-fields} \end{matrix}\right\} \cong \left\{\begin{matrix} H \leq \mathrm{Gal}^{\partial}(E/F) \\ \text{Zariski-closed} \\ \text{subgroups} \end{matrix}\right\}^{op}$$
>
> $$M \longmapsto \mathrm{Gal}^{\partial}(E/M)$$
> $$E^H \longleftarrow H$$