# §5. The geometry of Picard-Vessiot rings

Main goal: Show that $\partial$-Galois group is a linear algebraic group.

In this section, $F$ will denote a $\partial$-field with $\mathrm{char}(F) = 0$ and algebraically closed field of constants $K = K_F$.

**Proposition 5.1.** Let $A \in F^{n\times n}$ and $R$ Picard-Vessiot ring for $A$. Suppose $F \subseteq S$ $\partial$-ring extension such that

1. $S$ is an integral domain with $K_{\mathrm{Quot}(S)} = K$.
2. $\exists\ X \in \mathrm{GL}(n, S) : \partial(X) = AX$.

Denote by

$$U := (S \otimes_F R)^\partial$$

the $K$-algebra of constants in $S \otimes_F R$.

Then, the following hold:

(1) The map

$$\Theta : S \otimes_K U \longrightarrow S \otimes_F R, \quad s \otimes u \longmapsto (s \otimes 1) \cdot u$$

is an $S$-linear isomorphism of $\partial$-rings.

(2) Let $Y \in GL(n, R)$ generating fundamental matrix (i.e., $\partial(Y) = AY$ and $R = F\langle Y, Y^{-1}\rangle$). Then we have

$$U = K\langle Z, Z^{-1}\rangle$$

where $Z := X^{-1} \otimes Y \in GL(n, U)$.

(3) $U$ is a reduced $K$-algebra.

Proof. (1) Consider, as in (2), the matrix

$$Z := X^{-1} \otimes Y \in GL(n, U) \qquad (\partial(Z) = 0).$$

Then we have

$$(X \otimes I) \cdot (X^{-1} \otimes Y) = I \otimes Y.$$

$A \otimes B := (A \otimes 1) \cdot (1 \otimes B)$

and

$$(X^{-1} \otimes I)(X \otimes Y^{-1}) = I \otimes Y^{-1}$$

so that $\Theta$ is surjective, since $R = F\langle Y, Y^{-1}\rangle$.

To show that $\Theta$ is injective, we choose a $K$-basis $\{e_i\}$ of $U$.

Let

$$0 \neq f = \sum s_i \otimes_K e_i$$

of minimal length with $\Theta(f) = 0$. Wlog, we assume that $s := s_1 \neq 0$. Then we have

$$g := \partial(f)s - f \cdot \partial(s) \quad \in \operatorname{Ker}(\Theta)$$

of smaller length and thus $= 0$.

Therefore, for all $i$, we have

$$\partial(s_i) \cdot s - s_i \partial(s) = 0$$

so that, in $\operatorname{Quot}(S)$, we

$$\partial\left(\frac{s_i}{s}\right) = 0$$

and thus $\frac{s_i}{s} \in K$.

Thus, there exists $u \in U$ with

$$\frac{f}{s \otimes 1} = 1 \otimes u .$$

$$\Rightarrow \quad f = s \otimes u .$$

Since $f \in \operatorname{Ker}(\Theta)$, we have $(s \otimes 1) \cdot u = 0$.

But multiplication by $s$ is injective, since $S$ integral domain (and $s \neq 0$). It further follows(!) that multiplication by $s \in I$ remains injective on $S \otimes_K U$ (choose $K$-basis of $U$), so that $u = 0$. $\Rightarrow f = 0$ ↯.

(2) Consider the inclusion

$$i : K\langle z, z^{-1}\rangle \hookrightarrow U$$

and replace $S$ by $E := \mathrm{Quot}(S)$. Then the restriction $\Theta'$ of $\Theta$ to $E \otimes_K K\langle z, z^{-1}\rangle$ is injective and surjective (by the argument of (1)) so that $\Theta'$ is an isomorphism:

$$\begin{tikzcd}
E \otimes_K K\langle z, z^{-1}\rangle \arrow[r, "\Theta'", "\cong"'] \arrow[d, hook] \arrow[dr, phantom, "="] & E \otimes_F R \\
E \otimes_K U \arrow[ur, "\Theta", "\cong"'] &
\end{tikzcd}$$

must also be iso

$\Rightarrow$

$$E \otimes_K i : \quad E \otimes_K K\langle z, z^{-1}\rangle \xrightarrow{\cong} E \otimes_K U .$$

But then $i$ must also be surjective: the image of a $K$-basis of $K\langle z, z^{-1}\rangle$ is a $K$-basis of $U$ (since it an $E$-basis of $E \otimes_K U$).

(3) To show that $U$ is reduced, consider the isomorphism

$$\theta : S \otimes_K U \xrightarrow{\cong} S \otimes_F R$$

from (1). Both $R$ and $S$ are reduced by assumption (since integral domains) and therefore, by Lemma 5.2 below, $S \otimes_F R$ is reduced and thus $U \subseteq S \otimes_K U \cong S \otimes_F R$ is reduced. $\square$

**Lemma 5.2.** Suppose $F$ is a field of characteristic $0$ and $R_1, R_2$ are reduced $F$-algebras. Then $R_1 \otimes_F R_2$ is reduced.

Proof. See tutorial, also Lemma A.6 in Put-Singer.

**Corollary 5.3.** Let $E/F$ PV-extension for $A \in F^{n \times n}$ generated by fundamental matrix $Y \in GL(n, E)$. Then

$$F\langle Y, Y^{-1}\rangle \subseteq E$$

is a Picard-Vessiot ring for $A$.

Proof. Let $R$ PV-ring for $A$. By Prop 5.1, we obtain an $E$-linear isomorphism

$$\Theta : E \otimes_K U \xrightarrow{\cong} E \otimes_F R$$

We restrict the inverse of $\Theta$ to $R$ to obtain

$$i : R \hookrightarrow E \otimes_K U .$$

Let $m \trianglelefteq U$ be maximal ideal and $L = U/m$.

Since $U$ is finitely generated, we have

$$\begin{tikzcd} K[X_1, \dots, X_n] \arrow[r, two heads] \arrow[rr, bend right, "\pi"'] & U \arrow[r, two heads] & L = U/m . \end{tikzcd}$$

where $\ker(\pi) \trianglelefteq K[X_1, \dots X_n]$ maximal ideal, thus by Nullstellensatz of the form

$$(X_1 - \lambda_1, \dots, X_n - \lambda_n) .$$

Thus $L = K$. Now we form the composite

$$j : R \hookrightarrow E \otimes_K U \longrightarrow E \otimes_K \underbrace{U/m}_{K} \cong E$$

which is injective, since $R$ $\partial$-simple.

Suppose $R = F\langle X, X^{-1} \rangle$ with $\partial(X) = AX$. Then

$$j(X) = Y \cdot C \quad \text{with } C \in GL(n, K).$$

In particular, we have

$$j(R) = F\langle Y, Y^{-1}\rangle \qquad \square$$

**Cor 5.4** Let $R, R'$ PV-rings for $A \in F^{n\times n}$. Then the functor

$$\underline{\mathrm{Isom}}^\partial(R,R') : \mathrm{Calg}_K \longrightarrow \mathrm{Set},$$
$$L \longmapsto \underbrace{\mathrm{Isom}^\partial_{F\otimes_K L}(R\otimes_K L, R'\otimes_K L)}_{F\otimes_K L\text{-linear isos of } \partial\text{-rings}}$$

is represented by the $K$-algebra

$$U = (R' \otimes_F R)^\partial,$$

i.e.,

$$\underline{\mathrm{Isom}}^\partial(R,R') \cong \mathrm{Hom}_{\mathrm{Calg}_K}(U, -)$$

Proof. Let $L$ be a $K$-algebra. Then, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Calg}_K}(U, L) &\cong \mathrm{Hom}^\partial_{R'}(R'\otimes_K U, R'\otimes_K L) && (\mathrm{im.}) \\
&\cong \mathrm{Hom}^\partial_{R'}(R'\otimes_F R, R'\otimes_K L) && (5.1) \\
&\cong \mathrm{Hom}^\partial_F(R, R'\otimes_K L)
\end{aligned}$$

$\cong \mathrm{Hom}^{\partial}_{F\otimes_K L}(R\otimes_K L, R'\otimes_K L)$

$\cong \mathrm{Isom}^{\partial}_{F\otimes_K L}(R\otimes_K L, R'\otimes_K L)$

First note that any

$$f : R \longrightarrow R'\otimes_K L$$

is injective, since $R$ is $\partial$-simple ($\overset{(!)}{\Rightarrow}$ its extension to $R\otimes_K L$ remains injective). Also surjective, since both $\underset{X}{R}$ and $\underset{Y}{R'}$ are generated by fundamental matrices and $f(X) = Y\cdot C$ with $C \in GL(n, \underbrace{(R'\otimes_K L)^{\partial}}_{L})$ ✓. □.

> Cor 5.5. For every $R, R'$ PV rings for the same $A \in F^{n\times n}$, there exists an $F$-linear $\partial$-isom $R \cong R'$.

Proof: Have already shown statement directly, but also follows immediately from Cor 5.4 + Nullstellensatz.

Since

$$U = (R' \otimes_F R)^\partial$$

is finitely generated, choose maximal ideal $m \trianglelefteq U$, and obtain (Nullstellensatz)

$$\varphi : U \longrightarrow U/m = K \qquad \in \mathrm{Hom}_K(U, K)$$

which by 5.4 corresponds to a unique $K$-valued point in

$$\underline{\mathrm{Isom}}^\partial(R, R')(K) = \mathrm{Isom}^\partial_F(R, R')$$

which is the desired isomorphism. $\square$

**Cor 5.6.** Let $R/F$ Picard-Vessiot ring for $A \in F^{n \times n}$. Then, the $\partial$-Galois group

$$\mathrm{Aut}^\partial(E/F) = \mathrm{Aut}^\partial(R/F)$$

can be canonically identified with the group of $K$-valued points $\mathrm{Hom}_K(U, K)$ of a linear algebraic group $G(K)$ with coordinate ring $U = (R \otimes_F R)^\partial$. Further, the choice of a generating fundamental matrix $Y \in GL(n, R)$ exhibits $G(K)$ via the map $X \longmapsto z := Y^{-1} \otimes Y$

$$\underbrace{K[X, X^{-1}]}_{\mathcal{O}(GL(n,K))} \twoheadrightarrow \underbrace{U = K\langle z, z^{-1}\rangle}_{\mathcal{O}(G(K))}$$

as a Zariski-closed subgroup of $GL(n,K)$.

<u>Proof</u> From Cor 5.4, we have (with $R' = R$)

$$G(K) = \mathrm{Hom}_K(U, K) \cong \mathrm{Isom}^{\partial}_F(R, R) = \mathrm{Aut}^{\partial}(R/F).$$

setting $G := \underline{\mathrm{Isom}}^{\partial}(R, R)$, we obtain natural transformations

HW

$$\begin{aligned} m &: G \times G \to G \\ i &: G \to G \\ e &: pt \to G \end{aligned} \quad \rightsquigarrow \quad \underbrace{\begin{aligned} \mu &: U \to U \otimes_K U \\ \iota &: U \to U \\ \varepsilon &: U \to K \end{aligned}}_{\text{determines a group object in } (\mathrm{Calg}_K^{\mathrm{red, f.g.}})^{op} \simeq \mathrm{Var}_K}$$

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$\rightsquigarrow$ $G(K)$ is a group object in $\mathrm{Var}_K$

i.e., a linear algebraic group.

The remaining statements follow from observing that a $K$-valued point

$$\varphi : U \longrightarrow K \qquad \in \quad G(K)$$

yields a matrix

$$\varphi(Z) \in GL(n, K)$$

and the $\partial$-automorphism of $R$ that corresponds to $\varphi$ is given by

$$R \longrightarrow R, \; Y \longmapsto Y \cdot \varphi(Z). \qquad \square.$$