It turns out to be fruitful to slightly generalize the notion of an affine K-variety so that <u>all</u> commutative K-algebras arise as coordinate rings. These generalized varieties are called <u>affine schemes</u> and can, for example, be defined via

$$\left\{\begin{matrix}\text{affine } K\text{-}\\ \text{schemes}\end{matrix}\right\} := \mathrm{Calg}_K^{\mathrm{op}}.$$

<u>The Zariski topology.</u>

Let $n \in \mathbb{N}$. A subset

$$V \subseteq K^n$$

is called <u>(Zariski)closed</u> if

$$V = V(f_1, \dots, f_r) \qquad ; \; f_i \in K[X_1, \dots, X_n],$$

i.e., if V is an affine K-variety.

By Lemma 4.7, we have an inclusion-reversing bijection:

$$\left\{ \begin{matrix} V \in K^n \\ \text{Zariski-closed} \end{matrix} \right\} \cong \left\{ \begin{matrix} I \trianglelefteq K[X_1, \ldots, X_n] \text{ ideal} \\ \text{with } \sqrt{I} = I \end{matrix} \right\}$$

$$V \longmapsto I(V)$$

$$V(I) \longleftarrow I \quad .$$

Proposition 4.9. The Zariski-closed subsets form the closed subsets of a topology on K^n.

Proof We have $K^n = V(0)$ and $\emptyset = V(1)$.

Suppose $V = V(I)$ and $W = V(J)$ Z-closed.

Then we claim that

$$V \cup W = V(I \cap J) \quad .$$

Clear: $V \cup W \subseteq V(I \cap J)$.

Let $x \notin V \cup W$ then there exists $f \in I : f(x) \neq 0$ and $g \in J : g(x) \neq 0$. But then $f \cdot g(x) \neq 0$ with $f \cdot g \in I \cap J$. Thus $x \notin V(I \cap J)$.

Now suppose we are given $\{V_a\}_{a \in A}$ possibly infinite family of Z-closed subsets $V_a = V(I_a)$.

Then we have

$$\bigcap_{a\in A} V_a = V(\langle \bigcup_{a\in A} I_a \rangle)$$

where $\langle \bigcup_{a\in A} I_a\rangle$ is the smallest ideal containing $\bigcup_{a\in A} I_a$. Note that, by Basissatz,

$$\langle \bigcup_{a\in A} I_a\rangle = (f_1, \dots, f_r)$$

so that $\bigcap_{a\in A} V_a$ is indeed an affine variety. $\square$

The topology from Prop 4.4 is called the Zariski topology on k^n. Further, we define the Zariski topology on any subset $X \subseteq k^n$ to be the induced topology, i.e., $Y \subseteq X$ is closed iff $\exists\, V \subseteq k^n$ Zariski closed with $Y = V \cap X$. In particular, we obtain the Zariski topology on an affine variety V, and we have

$$\left\{ \begin{matrix} W \subseteq V \\ Z\text{-closed} \end{matrix} \right\} \cong \left\{ \begin{matrix} I(V) \subseteq J \trianglelefteq k[X_1, \dots, X_n] \\ \sqrt{J} = J \end{matrix} \right\}.$$

Generalized points.

Let $V \in \mathrm{Var}_K$ and $R \in \mathrm{Calg}_K$. An R-valued point of V is a homomorphism

$$\mathcal{O}(V) \longrightarrow R$$

of K-algebras. We denote by $V(R)$ the set of R-valued points of V.

Example 4.9. Let

$$V \subseteq K^n$$

affine K-variety. Then we have

$$\begin{aligned} V(K) &= \mathrm{Hom}_K(\mathcal{O}(V), K) \\ &\cong \mathrm{Hom}_K(K[X_1, \dots, X_n]/I(V), K) \\ &\cong \{(x_1, \dots, x_n) \in K^n \mid \forall f \in I(V) : f(\underline{x}) = 0\} \\ &= V. \end{aligned}$$

The point $u \in V(\mathcal{O}(V))$ corresponding to $\mathrm{id} : \mathcal{O}(V) \to \mathcal{O}(V)$ is called the universal point of V.

The k-algebra

$$k[\varepsilon] := k[X]/(X^2) \qquad \text{with } \varepsilon := \bar{X}$$

is called the <u>algebra of dual numbers</u>.

The elements of $k[\varepsilon]$ are thus of the form

$$f(\varepsilon) = f_0 + f_1\varepsilon \qquad \text{with } f_0, f_1 \in k.$$

We note the following homomorphisms (!) of k-algs:

- $\pi : k[\varepsilon] \to k, \; f(\varepsilon) \to f_0$
- $\alpha : k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon], \; (f(\varepsilon), g(\varepsilon)) \mapsto f_0 + (f_1 + g_1)\varepsilon$
- for every $\lambda \in k$:

  $\sigma_\lambda : k[\varepsilon] \to k[\varepsilon], \; f(\varepsilon) \mapsto f_0 + \lambda f_1 \varepsilon$

<u>Example 4.10.</u> Let $V \in \mathrm{Var}_k$. Consider the map

$$\pi_* : V(k[\varepsilon]) \longrightarrow V(k), \quad \varphi \longmapsto \pi \circ \varphi.$$

For a given k-valued point $x \in V(k)$, we define the <u>tangent space of V at x</u> to be

$$T_x V := \pi_*^{-1}(x).$$

Postcomposition with α and $\{\sigma_\lambda \mid \lambda \in k\}$ equips (HW6) the set T_xV with the structure of a k vector space.

Linear algebraic groups.

Let $\mathcal{C}$ be category with finite products. In particular, $\mathcal{C}$ has a final object denoted by $\mathbb{1}$.

We define a group object in $\mathcal{C}$ to be an object $G \in \mathcal{C}$ equipped with

- a unit morphism
$$e : \mathbb{1} \longrightarrow G$$
- a multiplication morphism
$$m : G \times G \longrightarrow G$$
- an inversion morphism
$$i : G \longrightarrow G$$

satisfying

1. associativity : $m \circ (m \times id) \overset{G\times G\times G \to G}{=} m \circ (id \times m)$
2. unitality : $m \circ (e \times id) = m(id \times e) = id$
3. inverses : $m \circ (id \times i) \circ d = id = m \circ (i \times id) \circ d$

   where $d : G \to G \times G$.

Example 4.11. A group object in Set is a group.

Definition 4.12. A linear algebraic group is a group object in Var_K.

Note that the equivalence $\mathrm{Var}_K \simeq (\mathrm{Calg}_K^{\mathrm{f.g.,red}})^{\mathrm{op}}$ allows us to equivalently describe a linear algebraic group G in terms of its coordinate ring $R := \mathcal{O}(G)$:

- counit: $\varepsilon : R \longrightarrow K$
- comultiplication: $\mu : R \longrightarrow R \otimes_K R$
- coinversion: $\imath : R \longrightarrow R$

in Calg_K satisfying the opposites of 1., 2., and 3.

Example 4.13. Let $G = GL(n,K)$ considered as an affine variety as in Example 4.2. In particular, the coordinate ring of $GL(n,K)$ is given by

$$\mathcal{O}(GL(n,K)) = K[x_{ij}, z] / (\det(x) z - 1) .$$

We observe that the maps

$$e : \mathbb{A}_K^0 \longrightarrow GL(n,K), \quad pt \longmapsto \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I$$

- $m: G \times G \longrightarrow G$

$$((x_{ij}, \det(x)^{-1}), (y_{ij}, \det(y)^{-1})) \longmapsto ((X \cdot Y)_{ij}, \det(XY)^{-1})$$

- $i: G \longrightarrow G,$

$$(x_{ij}, \det(x)^{-1}) \longmapsto ((\underbrace{X^{\#}}_{\text{adjunct matrix}} \cdot \det(x)^{-1})_{ij}, \det(x))$$

are defined by polynomial expressions so that they are morphisms in Var_K. This makes $GL(n,K)$ a linear algebraic group.

Example 4.14. The special linear group $SL(n,K)$ is a linear algebraic group. In fact, it is a Zariski-closed subgroup of $GL(n,K)$:

$$SL(n,K) \leq GL(n,K).$$

Derivations and Lie algebras

Let V be an affine k-variety with coordinate ring $R := \mathcal{O}(V)$.
We consider $R[\varepsilon] := R \otimes_k k[\varepsilon]$ and the map

$$\pi_* : V(R[\varepsilon]) \longrightarrow V(R)$$

on generalized points. Let $u \in V(R)$ be the universal point. We define

$$\mathrm{Vect}(V) := \text{``}T_u V\text{''} = \pi_*^{-1}(u)$$

to be the set of vector fields on V. Slightly generalizing Example 4.10, the set $\mathrm{Vect}(V)$ admits a k-vector space structure.

Let $\xi \in \mathrm{Vect}(V)$ vector field and $x \in V(k)$, then we obtain a tangent vector $\xi_x \in T_x V$ via

$$\xi_x : \begin{tikzcd} R \arrow[r, "\xi"] \arrow[rrd, "x"'] & R \otimes_k k[\varepsilon] \arrow[r, "x \otimes \mathrm{id}"] \arrow[rd, phantom, "="] & k[\varepsilon] \arrow[d, "\pi"] \\ & & k \end{tikzcd}$$

This explains the terminology "vector field".

Let $\mathrm{Der}_K(R)$ denote the set of K-linear derivations on R, i.e., K-linear maps $\delta : R \to R$ satisfying the Leibniz rule: $\delta(r\cdot s) = \delta(r)s + r\delta(s)$.

The set $\mathrm{Der}_K(R)$ forms a K-vector space via

$$(\lambda\delta)(r) = \lambda\,\delta(r) \quad ; \quad \lambda \in K$$

$$(\delta_1+\delta_2)(r) = \delta_1(r)+\delta_2(r) \quad , \quad \delta_i \in \mathrm{Der}_K(R).$$

Proposition 4.16. There is a canonical isomorphism

$$\mathrm{Vect}(V) \cong \mathrm{Der}_K(\mathcal{O}(V))$$

of K-vector spaces.

Proof Using the decomposition

$$R[\varepsilon] \cong R \oplus R\varepsilon$$

as R-modules, we may express $\xi \in \mathrm{Vect}(V)$

$$\xi = \mathrm{id} + \delta\cdot\varepsilon$$

for some map $\delta : R \to R$. The requirement that

$$\xi : R \longrightarrow R[\varepsilon]$$

is a homomorphism of K-algebras, is equivalent to δ being a derivation, e.g.

$$\xi(r\cdot s) = rs + \delta(rs)\cdot\varepsilon$$

$$\xi(r)\cdot\xi(s) = (r+\delta(r)\varepsilon)\cdot(s+\delta(s)\varepsilon)$$

$$= rs + (\delta(r)\cdot s + r\cdot\delta(s))\varepsilon \qquad \square$$

Let $\delta_1, \delta_2 \in \mathrm{Der}_K(R)$. Then the map

$$[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 : R \longrightarrow R$$

is again a K-linear derivation (!).

Prop. 4.17. The operation $[-,-]$ on $\mathrm{Der}_K(R)$ has the following properties:

1. $[-,-]$ is K-linear in each component,
2. $\forall x,y \in \mathrm{Der}_K(R): \ [x,y] = -[y,x]$
3. $\forall x,y,z \in \mathrm{Der}_K(R): \ [x,[y,z]] = [[x,y],z] + [y,[x,z]]$

Proof. Direct computation.

A K-vector space $\mathfrak{g}$, equipped with an operation

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying 1., 2., and 3. as in Prop. 4.17 is called a Lie algebra. We have thus constructed a Lie algebra structure on the K-vector space of vector fields on V / K-linear derivations of $\mathcal{O}(V)$.

Now suppose that is a linear algebraic group. Let $g, h \in G$ and $v \in T_h G$ be a tangent vector. Then, interpreting

$$g \in G = G(K) \hookrightarrow G(K[\varepsilon])$$

and using the multiplication map

$$\begin{aligned} G(K[\varepsilon]) \times G(K[\varepsilon]) &\longrightarrow G(K[\varepsilon]) \\ (g\ ,\ v) &\longmapsto g \cdot v \end{aligned}$$

we obtain a "translated" tangent vector $g \cdot v \overset{(!)}{\in} T_{gh} G$.

A vector field $\xi \in \mathrm{Vect}(G)$ is called left-invariant if

$$\forall g, h \in G : \xi_{g \cdot h} = g \xi_h .$$

Proposition 4.18. Let $\mathcal{L}(G) \subseteq \mathrm{Vect}(G)$ denote the set of left-invariant vector fields on G. Then

(1) The Lie bracket on $\mathrm{Vect}(G)$ restricts to $\mathcal{L}(G)$ so that $\mathcal{L}(G) \subseteq \mathrm{Vect}(G)$ becomes a sub Lie algebra.

(2) The map $\mathcal{L}(G) \longrightarrow T_eG$, $\xi \longmapsto \xi_e$ defines an isomorphism of k-vector spaces.

Proof (1) Under the isomorphism

$$\mathrm{Vect}(G) \cong \mathrm{Der}_k(R) \qquad ; \; R = \mathcal{O}(G).$$

the left invariance of $\xi = \mathrm{id} + \delta\varepsilon$ translates to the property (!)

$$\forall g, h \in G, \forall \varphi \in \mathcal{O}(G) : \; \delta(\varphi(g.-))(h) = \delta(\varphi)(gh)$$

This property is preserved under $[-,-]$ (!).

(2) We define an inverse to the map

$$\Phi : \mathcal{L}(G) \to T_eG, \; \xi \longmapsto \xi_e$$

as follows:

Let $R := \mathcal{O}(G)$ and let

$$u \in G(R) \hookrightarrow G(R[\varepsilon])$$

be the universal point of G ($u = id : R \to R$).

We then consider, for $v \in T_e G$, the multiplication

$$\begin{array}{ccccc} G(R[\varepsilon]) & \times & G(R[\varepsilon]) & \longrightarrow & G(R[\varepsilon]) \\ (u & , & v) & \longmapsto & u.v \end{array}$$

to obtain the vector field (!) $u.v \in \mathrm{Vect}(G)$ given by "universal translation".

By construction (!), we have $u.v \in \mathcal{L}(G)$, and define

$$\Psi : T_e G \longrightarrow \mathcal{L}(G), \quad v \longmapsto u.v .$$

It is then straightforward to verify that Φ and Ψ are inverse to one another. $\square$

By Prop 4.18, we thus arrive at the statement that the tangent space $\mathfrak{g} = T_e G$ to a linear algebraic group at the identity $e \in G$ admits a canonical Lie algebra structure.

This is an algebraic analog of the (perhaps familiar) analogous statement for Lie groups.

Side note: Sophus Lie also introduced Lie groups in his study of symmetries of the solutions of differential equations.

Examples 4.20.

(1) Let $G = GL(n,K)$ with coordinate ring

$$\mathcal{O}(GL(n,K)) = K[X_{ij}, Y] / (\det(X)Y - 1).$$

A R-valued point of $GL(n,K)$:

$$G(R) = \mathrm{Hom}_K(\mathcal{O}(GL(n,K)), R)$$
$$= GL(n,R)$$

For $R = K[\varepsilon]$, we obtain $X \in GL(n, K[\varepsilon])$

$$X = A + B \cdot \varepsilon \quad \text{with } A, B \in K^{n \times n}$$

such that $\det(X) \in K[\varepsilon]^\times$.

$$X \in T_e GL(n,K) \Leftrightarrow X = I + B\varepsilon.$$

Then $\det(X) = 1 + \varepsilon \operatorname{tr}(B)$ which is always a unit with inverse $1 - \varepsilon\, tr(B)$.

Therefore

$$T_e GL(n,K) \cong \{ I + B\varepsilon \mid B \in K^{n\times n} \} \cong K^{n \times n}.$$

$$= \mathfrak{gl}_n(K)$$

The Lie bracket is just given by the commutator of matrices

$$[B_1, B_2] = B_1 \circ B_2 - B_2 \circ B_1 . \qquad (*)$$

(2) For any Zariski closed subgroup $G \subseteq GL(n,K)$, we have

$$\mathfrak{g} = T_e G \subseteq T_e GL(n,K) = K^{n\times n} = \mathfrak{gl}_n(K)$$

with Lie bracket obtained by restriction of $(*)$.

For example: $G = SL(n,K)$, we get for R-valued points

$$G(R) = SL(n,R) .$$

Thus

$$T_e SL(n,K) = \{ I + B\varepsilon \mid \overbrace{\det(I + B\varepsilon)}^{1+\varepsilon tr(B)} = 1 \}$$

$$\cong \{ B \in K^{n\times n} \mid tr(B) = 0 \} \cong \mathfrak{sl}_n(K).$$