So far: F ∂-field, $\mathrm{char}(F) = 0$, $K_F = \overline{K_F}$

$$\underbrace{\ell \in F[\partial], \text{ or } A \in F^{n\times n}}$$

$$\rightsquigarrow$$

M $F[\partial]$-module

$$\rightsquigarrow$$

E/F Picard-Vessiot extension

$$\rightsquigarrow$$

$\mathrm{Gal}^{\partial}(M) := \mathrm{Aut}^{\partial}(E/F)$

∂-Galois group

§4. Affine varieties and linear algebraic groups

Let K be algebraically closed field.

Definition 4.1. An affine K-variety is a subset
$$V \subseteq K^n \quad ; n \in \mathbb{N}$$
such that there exist polynomials $f_1, \dots, f_r \in K[X_1, \dots, X_n]$ with $V = \{ \underline{x} \in K^n \mid \forall 1 \leq i \leq r : f_i(\underline{x}) = 0 \}$.

A morphism between affine K-varieties $V \subseteq K^n$ and $W \subseteq K^m$ is a map
$$\varphi : V \longrightarrow W$$
of sets such that there exist polynomials
$$F_1, \dots, F_m \in K[X_1, \dots, X_n]$$
with
$$\forall \underline{x} \in V \subseteq K^n : \varphi(\underline{x}) = (F_1(\underline{x}), \dots, F_m(\underline{x})) .$$
Equipped with this notion of morphisms, the collection of affine K-varieties organizes into a category, denoted by Var_K.

Examples 4.2.

(1) k^n itself forms an affine variety, called affine n-space, denoted $\mathbb{A}^n(k)$.

(2) The set of roots of a polynomial $f(X) \in k[X]$ is an affine variety $\{\alpha \in k \mid f(\alpha) = 0\} \subseteq k$.

(3) Equip k^{n^2+1} with the variables

$$\{X_{ij} \mid 1 \leq i,j \leq n\} \quad \text{and} \quad Y .$$

Then the set

$$V = \{(x_{ij}, y) \in k^{n^2+1} \mid \det(X) \cdot Y - 1 = 0\} \subseteq k^{n^2+1}$$

forms an affine variety. Note that we have a bijection $V \cong GL(n,k)$ by forgetting the Y-coordinate (which is uniquely determined as the inverse of $\det(X)$). Thus, we may consider the $GL(n,k)$ as an affine k-variety.

Let $V \subseteq K^n$ be an affine K-variety. Consider the set

$$\mathcal{O}(V) := \mathrm{Mor}(V, \mathbb{A}^1(K))$$

of morphisms from V to $\mathbb{A}^1(K)$ in Var_K, called the coordinate ring of V.

$\mathcal{O}(V)$ forms a commutative K-algebra via

$$\left.\begin{aligned} (f+g)(\underline{x}) &:= f(\underline{x}) + g(\underline{x}) \\ (f\cdot g)(\underline{x}) &:= f(\underline{x})\cdot g(\underline{x}) \\ (\lambda f)(\underline{x}) &:= \lambda(f(\underline{x})) \end{aligned}\right\} \quad \begin{array}{l} f, g \in \mathcal{O}(V) \\ \lambda \in K \end{array}$$

We further introduce the ideal (!)

$$I(V) := \{ f \in K[X_1, \dots, X_n] \mid \forall \underline{x} \in V : f(\underline{x}) = 0 \} \trianglelefteq K[X_1, \dots, X_n]$$

of polynomials vanishing on V.

Prop 4.3. There is an isomorphism

$$K[X_1, \dots, X_n] / I(V) \cong \mathcal{O}(V)$$

of K-algebras.

Proof. Every $f \in K[X_1, \dots, X_n]$ yields a morphism

$$\varphi_f : V \longrightarrow \mathbb{A}^1(K), \quad \underline{x} \longmapsto f(\underline{x}).$$

This defines a surjective homomorphism

$$K[X_1, \dots, X_n] \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}(V), \quad f \longmapsto \varphi_f$$

of K-algebras with kernel $I(V)$. $\square$

Note that, given morphism $\psi : V \longrightarrow W$ of affine varieties, we obtain a homomorphism

$$\psi^* : \mathcal{O}(W) \longrightarrow \mathcal{O}(V), \quad \varphi \longmapsto \varphi \circ \psi$$

of K-algebras.

We further observe that the coordinate ring $\mathcal{O}(V)$ of an affine variety has the following special properties:

(1) $\mathcal{O}(V)$ is a <u>finitely generated</u> K-algebra, since by Prop 4.3 it is generated by the coordinate functions $x_1, \dots, x_n$.

(2) $\mathcal{O}(V)$ is a reduced K-algebra, i.e., for all $\varphi \in \mathcal{O}(V)$ and $n \geq 1$, the implication

$$\varphi^n = 0 \implies \varphi = 0$$

holds.

Theorem 4.4.

(1) The assignments

$$V \longmapsto \mathcal{O}(V)$$

$$(\psi : V \to W) \longmapsto (\psi^* : \mathcal{O}(W) \to \mathcal{O}(V))$$

defines a functor

$$\mathcal{O} : \mathrm{Var}_K \longrightarrow (\mathrm{Calg}_K)^{op}.$$

(2) The functor $\mathcal{O}$ is fully faithful.

(3) The essential image of $\mathcal{O}$ consists of finitely generated, reduced K-algebras.

The proof relies on two famous theorems of D. Hilbert, which we state here without proof:

Theorem 4.5. (Basissatz) Let k field and let $I \trianglelefteq k[X_1, \ldots, X_n]$ ideal. Then there exist finitely many polynomials $f_1, \ldots, f_r \in I$ such that $I = (f_1, \ldots, f_r)$, i.e., I is finitely generated as a $k[X_1, \ldots, X_n]$-module.

Theorem 4.6. (Nullstellensatz) Let k be an algebraically closed field. Then the maximal ideals of the ring $k[X_1, \ldots, X_n]$ are precisely the ideals of the form

$$\mathfrak{m} = (X_1 - c_1, \ldots, X_n - c_n)$$

with $c_1, \ldots, c_n \in k$.

Proof of Theorem 4.4: Statement (1) is immediate.

To show (2), we need to show that the maps

$$(-)^* : \mathrm{Mor}(V, W) \longrightarrow \mathrm{Hom}_k(\mathcal{O}(W), \mathcal{O}(V)), \; \psi \longmapsto \psi^*$$

are bijective. To this end, we construct an inverse of $(-)^*$: Suppose $V \subseteq k^n$ and $W \subseteq k^m$ so that

$$\mathcal{O}(V) \cong k[X_1, \ldots, X_n]/I(V) \quad \text{and} \quad \mathcal{O}(W) \cong k[Y_1, \ldots, Y_m]/I(W).$$

Given a homomorphism $f: \mathcal{O}(W) \longrightarrow \mathcal{O}(V)$ we choose representatives $F_1, \dots, F_m$ in $K[X_1, \dots, X_n]$ of $f(Y_1), \dots, f(Y_m)$ and define

$$\psi_f : V \longrightarrow W, \quad \underline{x} \longmapsto (F_1(\underline{x}), \dots, F_m(\underline{x})).$$

Different choices $\widehat{F}_1, \dots, \widehat{F}_m$ differ by polynomials in $I(V)$ so they will lead to the same map ψ_f, showing that $f \mapsto \psi_f$ only depends on f.

It is now straightforward to verify that the map

$$\begin{aligned} \mathrm{Hom}_K(\mathcal{O}(W), \mathcal{O}(V)) &\longrightarrow \mathrm{Mor}(V, W) \\ f &\longmapsto \psi_f \end{aligned}$$

is inverse to $(-)^*$.

We now show (3): Let R finitely generated K-alg. We choose generators $r_1, \dots, r_n \in R$ and thus obtain a surjective homomorphism

$$\pi : K[X_1, \dots, X_n] \twoheadrightarrow R, \quad X_i \mapsto r_i$$

Let $I := \ker(\pi)$ which, by the Basissatz, is generated by a finite number of polynomials $f_1, \dots, f_r$.

We then define

$$V(I) := \{ \underline{x} \in K^n \mid \forall i : f_i(\underline{x}) = 0 \} \subseteq K^n .$$

We now claim that, for R reduced, we have

$$I(V(I)) \cong I$$

so that

$$\mathcal{O}(V(I)) \cong K[X_1, \dots, X_n] / I(V(I)) \cong R .$$

Lemma 4.7 below implies that

$$I(V(I)) = \sqrt{I} = \{ f \in K[\underline{X}] \mid f^m \in I \text{ for some } m \in \mathbb{N} \}.$$

But we have that

$$R \cong K[X_1, \dots, X_n] / I$$

is reduced, and thus $\sqrt{I} = I$. $\square$

Lemma 4.7. Let $I \subseteq K[X_1, \dots, X_n]$ ideal.

Then

$$I(V(I)) = \sqrt{I}$$

where

$$\sqrt{I} := \{ f \in K[X_1, \dots, X_n] \mid \exists m \geq 1 : f^m \in I \}$$

(called the radical of I)

<u>Proof</u> "$\supseteq$" is clear. To show "$\subseteq$" suppose that

$$I = (f_1, \dots, f_r) \qquad \text{(Basissatz)}.$$

and let $f \in I(V(I))$. We define the ideal

$$J = (f_1, \dots, f_r, fY - 1) \trianglelefteq k[X_1, \dots, X_n, Y].$$

Note that, for $g \in k[X_1, \dots, X_n, Y]$, we have

$$g(x_1, \dots, x_n, y) = 0 \iff g \in (X_1 - x_1, \dots, X_n - x_n, Y - y)$$

so that, further, for $(x_1, \dots, x_n, y) \in k^{n+1}$:

$$(x_1, \dots, x_n, y) \in V(J) \iff J \subseteq (X_1 - x_1, \dots, X_n - x_n, Y - y).$$

However, by construction, we have

$$V(J) = \emptyset$$

so that, for every $(x_1, \dots, x_n, y) \in k^{n+1}$, we have

$$J \not\subseteq (X_1 - x_1, \dots, X_n - x_n, Y - y).$$

By Nullstellensatz, this implies that J is not contained in <u>any</u> maximal ideal of $k[X_1, \dots, X_n, Y]$.

Thus J cannot be a proper ideal $\Rightarrow J = k[X_1, \dots, X_n, Y]$.
In particular, we have $1 \in J$ and thus there exist $g_i \in k[X_1, \dots, X_n, Y]$ with

$$g_1 f_1 + \dots + g_r f_r + g_{r+1}(fY - 1) = 1.$$

We evaluate this expression at

$$Y = \frac{1}{f}$$

and multiply the resulting equation by a suitable power f^m to clear denominators, resulting in a relation of the form

$$f^m = \tilde{g}_1 \cdot f_1 + \tilde{g}_2 f_2 + \dots + \tilde{g}_r f_r \in I.$$

where

$$\tilde{g}_i(X_1, \dots, X_n) = g_i(X_1, \dots, X_n, \tfrac{1}{f}) \cdot f^m \in k[X_1, \dots, X_n].$$

$\square$

Remark 4.8. By Theorem 4.4, we an equivalence of categories

$$\begin{aligned} \mathrm{Var}_k \quad &\simeq (\mathrm{Calg}_k^{\mathrm{fg,red}})^{\mathrm{op}} \\ V \quad &\longmapsto \mathcal{O}(V) \end{aligned}$$

It turns out to be fruitful to slightly generalize the notion of an affine k-variety so that <u>all</u> commutative k-algebras arise as coordinate rings. These generalized varieties are called <u>affine schemes</u> and can, for example, be defined via

$$\left\{\begin{matrix}\text{affine}\\ \text{schemes}\end{matrix}\right\} := \mathrm{Calg}_k^{\mathrm{op}}.$$