Theorem 3.8. Let F ∂-field with $\mathrm{char}(F) = 0$ and $K_F = \overline{K_F}$, and let $A \in F^{n \times n}$. Then

(1) There exists a PV ring for A.

(2) Given R_1, R_2 PV rings for A, there exists an F-linear isomorphism $R_1 \cong R_2$ of ∂-rings.

Proof. (1) Let

$$\mathcal{U} := \underbrace{K[X_{ij} \mid 1 \le i, j \le n]}_{\substack{\text{polynomial ring} \\ \text{in } X_{ij}}} \underbrace{[\det(X)^{-1}]}_{\substack{\text{localization at} \\ \{\det(X)^n \mid n \in \mathbb{N}\}}}$$

We equip $\mathcal{U}$ with the derivation given by the formula

$$\partial(X) := A \cdot X$$

extended to $\mathcal{U}$ via additivity, Leibniz rule, and quotient rule. This makes $\mathcal{U}$ a ∂-ring (!). Consider the set

$$\{ P \subsetneq \mathcal{U} \mid P \text{ proper } \partial\text{-ideal in } \mathcal{U} \}$$

partially ordered by inclusion. By Zorn's lemma this poset has a maximal element

$$P \subsetneq U .$$

Note that P is a maximal ∂-ideal, not necessarily a maximal ideal!

Then (!) the quotient ∂-ring

$$R := U/P$$

is ∂-simple. Denoting by

$$\pi : U \twoheadrightarrow R$$

the quotient ∂-homomorphism, we have

$$Y := \pi(X) \in GL(n, R)$$

and, by construction, we have

$$R = F\langle Y_{ij}, \det(Y)^{-1} \mid 1 \le i, j \le n \rangle = F\langle Y, Y^{-1} \rangle$$

and

$$\partial(Y) = A \cdot Y .$$

$\Rightarrow$ R is a PV ring for A.

(12) Given R_1, R_2 PV rings for A, we form the ∂-ring (!)

$$R_1 \otimes_F R_2$$

with $(r_1 \otimes r_2)(s_1 \otimes s_2) = r_1 s_1 \otimes r_2 s_2$, and

$\partial(r_1 \otimes r_2) = \partial(r_1) \otimes r_2 + r_1 \otimes \partial(r_2)$.

Let $P \subsetneq R_1 \otimes_F R_2$ maximal ∂-ideal and consider

$$\begin{tikzcd}
r_1 \mapsto r_1 \otimes 1 & & 1 \otimes r_2 \mapsfrom r_2 & \\
R_1 \arrow[r, hook] \arrow[dr, "\varphi"'] & R_1 \otimes_F R_2 \arrow[d, "\pi"] & R_2 \arrow[l, hook'] \arrow[dl, "\psi"] \\
& \underbrace{R_1 \otimes R_2 / P}_{=:R} &
\end{tikzcd}$$

Note that, since R_1, R_2 are ∂-simple, φ and ψ are injective (the kernel of a ∂-ring hom. is a ∂-ideal).

Further, let $Y_1 \in GL(n, R_1)$ and $Y_2 \in GL(n, R_2)$ be generating fundamental matrices.

Then, we have

$$\partial(\underbrace{\varphi(Y_1)^{-1}\psi(Y_2)}_{C}) = 0$$

Thus, we have

$$\psi(Y_2) = \varphi(Y_1)\cdot C \qquad (*)$$

with $C \in GL(n, K_{Quot(R)})$.

Lemma 3.5 $\Rightarrow K_{Quot(R)} = K_F$.

So, by $(*)$, $\Rightarrow \operatorname{im}\psi = \operatorname{im}\varphi$

Therefore, we have

$$R_1 \xrightarrow[\varphi]{\cong} \operatorname{im}\varphi = \operatorname{im}\psi \xleftarrow[\psi]{\cong} R_2 \qquad \square$$

Theorem 3.9. Let F ∂-field, $\operatorname{char}(F)=0$, $K_F = \overline{K_F}$.
Let E/F Picard-Vessiot extension for $A \in F^{n\times n}$, generated by the entries of a fundamental matrix $Y \in GL(n, E)$. Then

$$R := F\langle Y_{ij}, \det(Y)^{-1} \mid 1 \le i,j \le n\rangle \subseteq E$$

is a Picard-Vessiot ring.

Proof. Will be given later.

Remark 3.10. From Theorems 3.8 and 3.9, we deduce that any matrix $A \in F^{n\times n}$, $\mathrm{char}(F)=0$, $k_F=\overline{k_F}$, has a Picard–Vessiot extension which is unique up to F-linear ∂-isomorphism.

Remark 3.11. The above definition of PV rings / extensions can be formulated more intrinsically in terms a the $F[\partial]$-module M_A corresponding to A: Indeed, for another choice of basis of M_A, we obtain a matrix $B \in F^{n\times n}$ with

$$B = T^{-1}AT - T^{-1}\partial(T) \quad ; \quad T \in GL(n,F).$$

Let $Y \in GL(n,E)$ fundamental matrix for A, then

$$\partial(T^{-1}Y) = -T^{-1}\partial(T)T^{-1}Y + T^{-1}AY$$

so that $\underbrace{T^{-1}Y}_{Z}$ is a fundamental matrix for B.

Thus, we have

$$F< Y_{ij}, \det(Y)^{-1}> = F< Z_{ij}, \det(Z)^{-1}>$$

and

$$F(Y_{ij}) = F(Z_{ij}).$$

We may thus speak of PV rings / extensions associated to a given $F[\partial]$-module M of finite F-dimension.

Definition 3.12. Let F ∂-field, $\mathrm{char}(F)=0$, $K_F=\overline{K_F}$. Let M $F[\partial]$-module of finite F-dimension and let E/F a Picard-Vessiot extension for M. We define the ∂-Galoisgroup of E/F (of M) as

$$\mathrm{Gal}^{\partial}(M) = \mathrm{Gal}^{\partial}(E/F) := \mathrm{Aut}^{\partial}(E/F),$$

the group of $F[\partial]$-linear field automorphisms of E.

Let E/F Picard-Vessiot extension for M $F[\partial]$-module as above. Then the map

$$\begin{aligned} \mathrm{Gal}^{\partial}(E/F) \times \mathrm{Sol}_E(M) &\longrightarrow \mathrm{Sol}_E(M) \qquad (+) \\ (\sigma \quad , \quad \varphi) &\longmapsto \sigma \circ \varphi \end{aligned}$$

defines an action of $\mathrm{Gal}^{\partial}(E/F)$ on the K_E-vector space $\mathrm{Sol}_E(M)$ via K_E-linear maps. We fix the notation $K := K_F = K_E$.

Let $A \in F^{n\times n}$ corresponding to the choice of some F-basis of M. Then we have

$$E = F(Y_{ij}) \quad \text{where } Y \in GL(n, E)$$

with $\partial(Y) = AY$.

Then, for $\sigma \in \mathrm{Gal}^{\partial}(E/F)$, we have

$$\partial(\underbrace{Y^{-1}\sigma(Y)}_{C_\sigma}) = -Y^{-1}\partial(Y)Y^{-1}\sigma(Y) + Y^{-1}\sigma(\partial(Y))$$

$$= -Y^{-1}A\sigma(Y) + Y^{-1}A\sigma(Y) = 0.$$

Therefore, we have $C_\sigma \in GL(n, K)$ such that

$$\sigma(Y) = Y \cdot C_\sigma. \qquad (*)$$

We obtain a homomorphism

$$\rho : \mathrm{Gal}^{\partial}(E/F) \longrightarrow GL(n,K),$$

$$\sigma \longmapsto C_\sigma$$

which corresponds to the action (+) with respect to the K-basis of $\mathrm{Sol}_E(M)$ given by the columns of Y. Since E/F generated by the entries of Y, it is further clear that the action (+) is faithful, i.e., ρ is injective.

In other words, via ρ, we may identify
$$\mathrm{Gal}^{\partial}(E/F) \leq GL(n,K)$$
with a subgroup of $GL(n,K)$.
Note, that this is analogous to the fact from ordinary Galois theory that the Galois group of a polynomial f can, upon enumerating the roots of f by $\alpha_1, \ldots, \alpha_n$, be identified with a subgroup of S_n.

Remark 3.13. The above definition of $\mathrm{Gal}^{\partial}(E/F)$ turns out to be too crude to obtain a reasonable Galois theory. As will be explained later, we also need to keep track of the fact that
$$\mathrm{Gal}^{\partial}(E/F) \leq GL(n,K)$$
is, beyond just being a subgroup, the zero locus of polynomials in $X_{i,j}$, $\det(X)^{-1}$ parametrizing the entries of matrices in $GL(n,K)$. This will be the subject of the next section.

Example 3.14. Let F be as above and consider the ∂-equation

$$\partial \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}}^{A} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad (*)$$

$K = k_F$

where $f \in F$. Assume that there does not exist $y \in F : \partial(y) = f$. On HW3, we have shown that a PV extension for A is given by

$$E = F(T)$$

with $\partial(T) = f$. E is generated by the entries of the fundamental solution matrix

$$Y = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \in GL(2, E).$$

An automorphism $\sigma \in Gal^{\partial}(E/F)$ is uniquely determined by $\sigma(T)$. Since $\partial(T) = f$, we have $\partial(\sigma(T)) = f$ so that $\lambda_\sigma := \sigma(T) - T \in K$.

Vice versa, for every $\lambda \in K$, the map

$$F(T) \longrightarrow F(T), \quad g(T) \longmapsto g(T + \lambda)$$

defines an F-linear ∂-automorphism.

Thus, we have

$$\mathrm{Gal}^{\partial}(E/F) \cong (K,+)$$
$$\sigma \longmapsto \lambda_\sigma$$

The choice of the fundamental matrix Y gives rise to the homomorphism

$$\rho: \mathrm{Gal}^{\partial}(E/F) \longrightarrow GL(2,K)$$
$$\sigma \longmapsto C_\sigma = \begin{pmatrix} 1 & \lambda_\sigma \\ 0 & 1 \end{pmatrix}$$

Thus

$$\mathrm{Gal}^{\partial}(E/F) \cong \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \,\middle|\, \lambda \in K \right\} \leq GL(2,K).$$
$$\|$$
$$\left\{ X \in GL(2,K) \,\middle|\, \begin{matrix} X_{11} - 1 = 0 \\ X_{21} = 0 \end{matrix} ; X_{22} - 1 = 0 \right\}.$$

Example 3.15. Let F as above and consider the ∂-equation

$$\partial(v) = \underbrace{a}_{A=(a)} v \qquad (*)$$

with $a \in F^*$.

Assume that

$$\partial(y) = nay \quad ; \quad n \geq 1$$

does not have nonzero solutions in F.

Then, on HW3, we have seen that

$$E = F(T) \quad ; \quad \partial(T) = aT$$

is a Picard-Vessiot extension generated by the fundamental matrix $Y = (T)$.

Then, for $\sigma \in Gal^{\partial}(E/F)$, we have

$$\sigma(T) = T \cdot \lambda_\sigma \quad ; \quad \lambda_\sigma \in K^*$$

and further

$$Gal^{\partial}(E/F) \cong (K^*, \cdot) = GL(1, K)$$

$$\sigma \longmapsto \lambda_\sigma$$