§2. Basic differential-algebraic notions

Convention: Unless otherwise stated, all rings are assumed to be commutative and unital.

Let R be a ring. A derivation on R is a map

$$\partial : R \longrightarrow R$$

satisfying, $\forall x, y \in R$:

1. $\partial(x+y) = \partial(x) + \partial(y)$
2. $\partial(xy) = \partial(x)y + x\partial(y)$.

A ∂-ring ("differential ring") is a ring, equipped with a derivation. A ∂-ring (R, ∂) is called a ∂-field if R is a field.

Examples 2.1 K field.

(1) $R = K[t]$, $\partial = \frac{d}{dt}$

(2) $F = K(t)$, $\partial = \frac{d}{dt}$

(3) $R = K[t_1, \dots, t_n]$, $\partial = \overbrace{f_1 \frac{d}{dt_1} + \dots + f_n \frac{d}{dt_n}}^{f_i \in R}$

A ∂-module over a ∂-ring (R, ∂) consists of

- an R-module M
- a map $\partial : M \longrightarrow M$

such that

1. $\partial(x+y) = \partial(x) + \partial(y)$
2. $\partial(r.m) = \partial_R(r).m + r.\partial(m)$.

Examples 2.2.

(1) Every ∂-ring is a ∂-module over itself.

(2) Any ideal

$$I \trianglelefteq R$$

in a ∂-ring (R, ∂) such that $\partial(I) \subseteq I$ defines a ∂-module, called ∂-ideal

A ∂-operator ℓ over a ∂-ring (R, ∂) is a polynomial in ∂ with coefficients in R:

$$\ell = r_n \partial^n + \ldots + r_0 \quad \in R[\partial]$$

The set $R[\partial]$ of ∂-operators over (R,∂) forms a noncommutative ring via the commutation rule

$$\partial . r = \partial(r) + r.\partial .$$

<u>Observation</u>(!) Given a ∂-ring (R,∂) we have

$$\left\{ \begin{array}{l} \partial\text{-modules} \\ \text{over } (R,\partial) \end{array} \right\} \sim \left\{ \begin{array}{l} \text{(left)} \\ R[\partial]\text{-modules} \end{array} \right\}$$

<u>Remark 2.3</u> Any ∂-operator ℓ over (R,∂) defines an ideal

$$R[\partial].\ell \subseteq R[\partial]$$

left-ideal and further a quotient $R[\partial]$-module

$$M := R[\partial]/R[\partial]\ell .$$

Let F be a ∂-field and let M be a $F[\partial]$-module. Let further $F \subseteq E$ an embedding of ∂-fields. In particular, E is an $F[\partial]$-module. We define

$$Sol_E(M) := Hom_{F[\partial]}(M, E).$$

Denote
$$K_E := \{ e \in E \mid \partial(e) = 0 \} \subseteq E$$
the subfield (!) of constants.

Observation (!) The set $\mathrm{Sol}_E(M)$ forms a K_E-vector space.

We call $\mathrm{Sol}_E(M)$ the vector space of solutions of M in E.

Example 2.4 Let $\ell \in F[\partial]$ with associated module $M = F[\partial]/F[\partial]\ell$. Then we have a canonical bijection
$$\begin{array}{ccc} \mathrm{Sol}_E(M) & \cong & \{ y \in E \mid \ell(y) = 0 \} \\ \varphi & \longmapsto & \varphi(1) \end{array}$$
exhibits $\mathrm{Sol}_E(M)$ as the K_E-vector space of solutions of ℓ in E.

More generally, let $F \subseteq E$ embedding of ∂-fields and M $F[\partial]$-module such that $\dim_F(M) = n < \infty$. Let $(b_1, \dots, b_n)$ be an F-basis of M. Then, expressing $\partial(b_i)$ in terms of $\underline{b}$, we obtain a matrix $A \in F^{n \times n}$

with

$$\partial\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = A\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Then, we have an identification

$$\mathrm{Sol}_E(\mathcal{M}) \cong \{ v \in E^n \mid \partial(v) = A(v) \}$$

with the K_E-vector space of solutions in E of the system

$$\left.\begin{aligned} \partial(v_1) &= a_{11} v_1 + \dots + a_{1n} v_n \\ &\vdots \\ \partial(v_n) &= a_{n1} v_1 + \dots + a_{nn} v_n \end{aligned}\right\} \quad (*)$$

of 1st order ∂-equations.

Vice versa, any such system of 1st order ∂-equations defines a ∂-module structure on F^n.

Conclusion An $F[\partial]$-module $\mathcal{M}$ with $\dim_F \mathcal{M} < \infty$ amounts to a coordinate-free description of a system of 1st order equations with $\mathrm{Sol}_E(\mathcal{M})$ as an intrinsic K_E-vector space of solutions.

Prop 2.5. Let $F \subseteq E$ embedding of ∂-fields, let M a $F[\partial]$-module with $\dim_F M = n < \infty$.

Then, we have

$$\dim_{K_E} Sol_E(M) \leq n$$

Proof. Note that we have

$$Sol_E(M) = \underbrace{Hom_{F[\partial]}(M,E)}_{K_E\text{-vs}} \subseteq \underbrace{Hom_F(M,E)}_{E\text{-vs of (!) } \dim_E = n}$$

, even $E[\partial]$-module

where, for $\varphi \in Hom_F(M,E)$, we set

$$\partial(\varphi) := \partial_E \circ \varphi - \varphi \circ \partial_M .$$

We have, for $\varphi \in Hom_F(M,E)$:

$$\varphi \in Sol_E(M) \iff \partial(\varphi) = 0$$

We will now show, for $\varphi_1, \dots, \varphi_r \in Sol_E(M) \subseteq \overbrace{Hom_F(M,E)}^{\dim_E = n}$:

$\{\varphi_i\}$ E-linearly dependent (in $Hom_F(M,E)$)

$\Rightarrow \{\varphi_i\}$ K_E-linearly dependent (in $Sol_E(M)$).

To this end, let

$$e_1\varphi_1 + \dots + e_r\varphi_r = 0 \qquad (*)$$

an E-linear relation where wlog, we assume

1. $e_1 = 1 \overset{(!)}{\rightsquigarrow} \partial(e_1) = 0$

2. $\varphi_2, \dots, \varphi_r$ are E-linearly independent.

Applying ∂ to $(*)$, we obtain

$$\partial(e_2)\varphi_2 + \dots + \partial(e_r)\varphi_r = 0$$

$\overset{2.}{\Rightarrow} \forall i : \partial(e_i) = 0 \Rightarrow \forall i : e_i \in K_E$ $\square$

Remark 2.6. Above, we explained how to construct $F[\partial]$-modules from ∂-operators and systems of 1st order equations:

1. $\ell = \partial^n + f_{n-1}\partial^{n-1} + \dots + f_0 \in F[\partial]$

$\rightsquigarrow M_\ell := F[\partial] / F[\partial]\ell$.

2. $\partial \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$; $A \in F^{n \times n}$

$\rightsquigarrow M_A := (F^n, \partial_A)$

Prop 2.5 then implies, for $F \subseteq E$, that the solution spaces

$$Sol_E(\ell) := Sol_E(M_\ell) \quad \text{and} \quad Sol_E(A) := Sol_E(M_A)$$

are K_E-vector spaces of dimension $\leq n$.

Interlude. The Wronskian

For elements $y_1, \dots, y_n$ of a ∂-field F, we define the Wronskian (determinant) of $\{y_i\}$ as

$$\operatorname{wr}(y_1, \dots, y_n) = \det \begin{pmatrix} y_1 & \cdots & y_n \\ \partial(y_1) & \cdots & \partial(y_n) \\ \vdots & & \\ \partial^{n-1}(y_1) & \cdots & \partial^{n-1}(y_n) \end{pmatrix} \in F .$$

Prop 2.7. Let F ∂-field and let $y_1, \dots, y_n \in F$.
Then

$y_1, \dots, y_n$ K_F-linearly dependent $\iff$ $\operatorname{wr}(y_1, \dots, y_n) = 0$.

Proof. "$\Rightarrow$" If $\sum \lambda_i y_i = 0$ with $\lambda_i \in K_F$,
then also $\forall k: \sum \lambda_i \partial^k(y_i) = 0$.
$\Rightarrow$ The columns of the Wronskian matrix are K_F-linearly dependent $\Rightarrow \operatorname{wr}(y_1, \dots, y_n) = 0$.

"$\Leftarrow$": Below, we will construct an ∂-operator

$$\ell \in F[\partial] \qquad (*)$$

of order n, such that $\forall i : \ell(y_i) = 0$.
Assuming ℓ has been constructed, we conclude as follows:

Each column vector $(y_i, \partial(y_i), \dots, \partial^{n-1}(y_i))^{tr}$ defines an $F[\partial]$-linear map

$$\varphi_i : F[\partial]/F[\partial]\ell \longrightarrow F,$$
$$\partial^k \longmapsto \partial^k(y_i)$$

i.e., $\varphi_i \in Sol_E(M_\ell)$. Further, the condition

$$wr(y_1, \dots, y_n) = 0$$

implies that $\{\varphi_i\}$ are F-linearly dependent in $Hom_F(M_\ell, F)$. By the proof of Prop 2.5, it follows that $\{\varphi_i\}$ are K_F-linearly dependent, in particular $\{\varphi_i(1) = y_i\}$ are K_F-linearly dependent.

It remains to construct the ∂-operator from (*).
We achieve this by induction on n:

<u>$n=1$</u>:

$$\ell = \begin{cases} \partial - \frac{\partial(y_1)}{y_1} \cdot \mathrm{id} & (\text{if } y_1 \neq 0) \\ \partial & (\text{if } y_1 = 0) \end{cases}$$

<u>$n-1 \rightsquigarrow n$</u>: Suppose $\ell_{n-1} \in F[\partial]$ of order $n-1$
with $\forall\, i = 1, \ldots, n-1$: $\ell_{n-1}(y_i)$.

We set

$$\ell_n := \begin{cases} \partial \ell_{n-1} - \frac{\partial \ell_{n-1}(y_n)}{\ell_{n-1}(y_n)} \ell_{n-1} & (\text{if } \ell_{n-1}(y_{n-1}) \neq 0) \\ \partial \ell_{n-1} & (\text{else}). \end{cases}$$

This concludes the proof. □

Remark 2.8. The statement of Prop 2.7 should be regarded as a ∂-analog of the statement that, for elements $x_1, \ldots, x_n$ in a field K, we have

$$\{x_i\} \text{ are not pairwise distinct} \iff \det \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & & x_n^{n-1} \end{pmatrix} = 0$$