§9. Elementary extensions

In this section, we provide an application of ∂-Galois theory to a classical question:

"Can the integral of $\exp(-x^2)$ be expressed as an elementary function?"

To answer the question, we need to say precisely what we mean by an "elementary function":

Definition 9.1. An elementary extension of a ∂-field F is a ∂-field extension of the form

$$L = M(t_1, \dots, t_n)$$

with $K_L = K_F$ and

1. $[M : F]$ is finite
2. For $1 \le i \le n$, we have either of:
   a) $\partial(t_i) \in M$
   b) $\partial(t_i)/t_i \in M$.

An element of an elementary extension of F is called elementary over F.

Theorem 9.2. Let E/F be a Picard-Vessiot extension with ∂-Galois group $G = \mathrm{Gal}^{\partial}(E/F)$. Suppose that E is contained in an elementary extension. Then G° is abelian.

For the proof, we will use the following proposition:

Proposition 9.3. Let E/F PV-extension with ∂-Galois group G.

(1) Suppose $G \cong G_1 \times G_2$. Then E is the composite of the Picard-Vessiot extensions $E_1 = E^{G_1}$ and $E_2 = E^{G_2}$.

(2) Suppose $F \subseteq E_1 \subseteq E$ and $F \subseteq E_2 \subseteq E$ are Picard-Vessiot extesics with $E = F(E_1, E_2)$. Then G is isomorphic to a closed subgroup of $\mathrm{Gal}^{\partial}(E_1/F) \times \mathrm{Gal}^{\partial}(E_2/F)$.

Proof (1) Let $E' = F(E_1, E_2)$ composite of E_1, E_2. Then, since $F \subseteq E_1, E_2 \subseteq E$, we have

1. $\mathrm{Gal}^{\partial}(E/E') \leq \mathrm{Gal}^{\partial}(E/E_1) = G_1 \cong G_1 \times \{e\}$
2. $\mathrm{Gal}^{\partial}(E/E') \leq \mathrm{Gal}^{\partial}(E/E_2) = G_2 \cong \{e\} \times G_2$

$\Rightarrow \mathrm{Gal}^{\partial}(E/E') \subseteq (G_1 \times \{e\}) \cap (\{e\} \times G_2) = \{e\} \times \{e\}$.

∂-Galois correspondence

$$\Longrightarrow \qquad E' = E.$$

(2) Let $Y_1 \in GL(n, E_1)$ and $Y_2 \in GL(m, E_2)$ be generating fundamental matrices. Then, since $E = F(E_1, E_2)$, the matrix

$$Y = \left(\begin{array}{c|c} Y_1 & 0 \\ \hline 0 & Y_2 \end{array}\right) \in GL(n+m, E)$$

is fundametal matrix with $E = F(Y_{ij})$.

These fund. matrices yield embeddings

$$\mathrm{Gal}^{\partial}(E_1/F) \leq GL(n,k)$$
$$\mathrm{Gal}^{\partial}(E_2/F) \leq GL(m,k)$$
$$\mathrm{Gal}^{\partial}(E/F) \leq GL(n+m,k)$$

In terms of these representations, we analyze the restriction map

$$\begin{array}{cccc} \rho : \mathrm{Gal}^{\partial}(E/F) & \longrightarrow & \mathrm{Gal}^{\partial}(E_1/F) \times \mathrm{Gal}^{\partial}(E_2/F) \\ \sigma & \longmapsto & \sigma|E_1 \qquad \sigma|E_2 \\ C_\sigma & & C_\sigma^{(1)} \qquad C_\sigma^{(2)} \end{array}$$

Have

$$\sigma(Y) = \begin{pmatrix} \sigma(Y_1) & 0 \\ 0 & \sigma(Y_2) \end{pmatrix} = \begin{pmatrix} Y_1 \cdot C_\sigma^{(1)} & 0 \\ 0 & Y_2 \cdot C_\sigma^{(2)} \end{pmatrix}$$

so that

$$C_\sigma = \begin{pmatrix} C_\sigma^{(1)} & 0 \\ 0 & C_\sigma^{(2)} \end{pmatrix}$$

and ρ is given via

$$C_\sigma \longmapsto (C_\sigma^{(1)}, C_\sigma^{(2)}) .$$

In particular, we have

$$\begin{tikzcd} \mathrm{Gal}^{\partial}(E/F) \arrow[r, hook, "\text{"} \Rightarrow \text{" closed}"] \arrow[dr, "\text{closed}"'] & \mathrm{Gal}^{\partial}(E_1/F) \times \mathrm{Gal}^{\partial}(E_2/F) \arrow[d, hook, "\text{closed}"] \\ & GL(n+m, K) \end{tikzcd}$$

$\square$

Proof of Theorem 9.2.

We have

$$\begin{array}{l} E \\ | \\ E^{G^0} \\ | \\ F \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{l} E \\ | \\ E \end{array}}\right\} \text{pure transcendental} \\ \left.\vphantom{\begin{array}{l} E \\ | \\ F \end{array}}\right\} \text{algebraic} \end{array}$$

replace F by E^{G^0}

thus, we may assume $G = G^0$. Thus, we have that E is contained in an elementary extension of the form $F(t_1, \ldots, t_n)$ with t_i as in Definition 9.1. But then

$$F(t_1, \ldots, t_n) \,/\, F$$

is a Picard-Vessiot extension $/ F$.

We have $\underbrace{\underbrace{F \subseteq E}_{G} \subseteq \underbrace{F(t_1, \ldots, t_n)}_{H}}_{\tilde{G}}$

with $G \cong \tilde{G}/H$. So it suffices to show $\tilde{G}$ abelian $\Rightarrow G$ abelian.

Thus, we may assume that

$$E = F(t_1, \dots, t_n)$$

and will show $\mathrm{Gal}^\partial(E/F)$ abelian:

We have, by HW3, we have

$$\mathrm{Gal}^\partial(F(t_i)/F) \cong \begin{cases} \mathbb{G}_a \\ \mathbb{G}_m \end{cases}$$

and by Prop. 9.3, we have

$$\mathrm{Gal}^\partial(E/F) \hookrightarrow \prod_{1 \le i \le n} \mathrm{Gal}^\partial(F(t_i)/F) \quad \text{abelian.}$$

$\square$

Theorem 9.4. The function $\int \exp(-x^2)$ is not elementary over $K(x)$.

Proof $\quad f := \int \exp(-x^2)$ is a solution of a ∂-equation over $(K(x), \partial(x) = 1)$.

$$f' = \exp(-x^2) = w$$

$$w' = -2x \cdot w$$

so that f is a solution of

the ∂-equation

$$y'' + 2x \cdot y' = 0$$

which translates to the system

$$\partial \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -2x \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix} \qquad (*) .$$

We may express f as a power series in $K[[x]]$ so that, we obtain a fundamental solution matrix

$$Y = \begin{pmatrix} 1 & f \\ 0 & w \end{pmatrix} \in GL(2, K((x)))$$

so that with $F = K(x)$, we have that

$$E = F(f) \subseteq K((x)) \quad \partial\text{-field}$$

is a Picard-Vessiot extension $/ F$.

By Theorem 9.2, it suffices to show that $\mathrm{Gal}^{\partial}(E/F)^{\circ}$ is not abelian.

We have a tower of ∂-field extensions:

$$F \overset{\substack{\partial(w) = -2x\cdot w}}{\subseteq} M = F(w) \overset{\substack{\partial(f) = w \in M}}{\subseteq} E = M(f).$$

We will now show:

1. $\mathrm{Gal}^{\partial}(M/F) \cong \mathbb{G}_m$.
2. $\mathrm{Gal}^{\partial}(E/M) \cong \mathbb{G}_a$.

By HW 7.3, we thus need to show

1. For every $n \in \mathbb{N} \setminus \{0\}$, the equation
$$\partial(y) = -2x\cdot n y \qquad (*)$$
does not have a nontrivial solution in F.

2. The equation
$$\partial(y) = w \qquad (+)$$
does not have a solution in M.

1. Suppose y is solution of $(*)$. Then we write
$$y = \frac{p}{q} \quad \text{with } p,q \in K[x],\ p \neq 0.$$
and with $\deg(p)$ minimal.

Then we differentiate $p = y \cdot q$ to obtain

$$p' = -2x \cdot n \cdot y q + y \cdot q'$$

so that

$$y = \frac{p'}{(q' - 2x \cdot n q)} \quad \text{with } \deg(p') < \deg(p) \text{ } \lightning.$$

(2.) Suppose (+) has a solution

$$y = \frac{p(w)}{q(w)} \quad \text{with } p, q \in F[w]$$

with leading coefficient of q is 1.

Differentiating $q y = p$ yields

$$\partial(p) = \partial(q) \cdot y + q w$$

$$\boxed{p = \lambda \cdot w^n + \dots}$$

multiplying by q:

$$\partial(p) q - p \partial(q) = q^2 w \qquad (*) .$$

Suppose $\deg_w(p) = n$ and $\deg_w(q) = m$, then

$$\partial(p) = \lambda' w^n + \lambda \cdot n(-2x) w^n + \text{lower}(w)$$

$$\partial(q) = m(-2x) w^m + \text{lower}(w)$$

$$q^2 w = w^{2m+1} + \text{lower}(w)$$

Plugging into (*) we have:

$$(\lambda' + \lambda\cdot(-2x)(n-m))\,w^{n+m} + \text{l.t.}$$
$$= w^{2m+1} + \text{l.t.}$$

This implies that either

1. $n > m+1$ and $\lambda' + \lambda(-2x)(n-m) = 0$ $\quad ↯$ 1.
2. $n = m+1$ and $\lambda' - 2x\lambda \overset{(*)}{=} 1$; $\lambda \in F$.

Suppose $\lambda = \frac{r}{s}$ with $r, s \in K[x]$ coprime.

Then differentiate $s\lambda = r$ to obtain

$$r' = s'\cdot\lambda + s(2x\lambda + 1)$$
$$\Leftrightarrow (r' - s) = (s' - 2xs)\lambda$$
$$\overset{\lambda = \frac{r}{s}}{\Rightarrow} (r'-s)s = (s'-2xs)r$$
$$\Rightarrow s \mid s' - 2xs \quad \Rightarrow \quad s \mid s' \Rightarrow s' = 0$$
$$\Rightarrow \lambda \in K[x]$$

But clearly, $(*)$ does not have polynomial solutions (compare leading terms in $(*)$).

Thus, we have

$$\left.\begin{array}{l} G_a\left\{\begin{array}{c}E\\ |\\ M\end{array}\right. \\ G_m\left\{\begin{array}{c} \\ |\\ F\end{array}\right.\end{array}\right\} G \qquad \Longrightarrow \qquad G/G_a \cong G_m$$

Consider representation $G \leq GL(2,K)$ in terms of the fundamental matrix

$$Y = \begin{pmatrix} 1 & f \\ 0 & w \end{pmatrix} \in GL(2,E)$$

$$\sigma(Y) = Y \cdot C_\sigma .$$

Since $\sigma\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, we see that $C_\sigma = \begin{pmatrix} 1 & c \\ 0 & \lambda \end{pmatrix}$.

$$G \hookrightarrow \left\{ \begin{pmatrix} 1 & c \\ 0 & \lambda \end{pmatrix} \,\middle|\, c \in K,\ \lambda \in K^\times \right\} \leq GL(2,K).$$

But we have $G_a \cong \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \,\middle|\, c \in K \right\} \subseteq G$.

$$\mathrm{Gal}^\partial(E/M) \leq \mathrm{Gal}^\partial(E/F)$$

$$f \mapsto f + c .$$

But further, we have

$$K^\times = \mathbb{G}_m$$

$$\begin{array}{ccc} \mathrm{Gal}^\partial(E/F) & \twoheadrightarrow & \underset{\parallel}{\mathrm{Gal}^\partial(H/F)} \\ \begin{pmatrix} 1 & c \\ 0 & \lambda \end{pmatrix} & \longmapsto & \{ w \mapsto \lambda w \} . \end{array}$$

(!)

$$\Rightarrow G \cong \left\{ \begin{pmatrix} 1 & c \\ 0 & \lambda \end{pmatrix} \,\middle|\, c \in K ,\ \lambda \in K^\times \right\} .$$

As affine variety $G \cong \mathbb{A}^1 \times \mathbb{A}^1 \setminus \{0\}$ connected, so that $G = G^0$.

Further, we have

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda c \\ 0 & \lambda \end{pmatrix}$$

$$\neq \qquad c \neq 0, \ \lambda \neq 1 .$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & \lambda \end{pmatrix}$$

$\Rightarrow G^0$ is not abelian.

In fact: $G^0 = \mathbb{G}_a \rtimes \mathbb{G}_m$.

$\Rightarrow f$ is not an elementary function. $\square$