<u>Last time</u> : E/F PV-ring with $G = \mathrm{Gal}^{\partial}(E/F)$

$$G\left\{\begin{array}{c} E \\ | \\ E^H \\ | \\ F \end{array}\right. \qquad \begin{array}{l} \left.\begin{array}{c} E \\ | \\ E^H \end{array}\right\} H \\ \left.\begin{array}{c} E^H \\ | \\ F \end{array}\right\} G/H \end{array}$$

Z-closed

$H \leq G$

normal

§8. Liouville extensions

Recall. Historically, the motivation for introducing Galois theory was the relation between group-theoretic properties of $G(f)$, $f \in K[x]$, and algebraic properties of the roots of f:

$$G(f) \text{ solvable } \Leftrightarrow \text{The roots of } f \text{ admit radical expressions.} \quad (*)$$

Are there analogous statements in ∂-Galois theory?

Definition 8.1. F ∂-field with $K = F^{\partial}$ alg. closed and char$(F) = 0$. An extension M/F of ∂-fields is called Liouvillian if $M^{\partial} = K$ and there exists a tower of ∂-field extensions

$$F = M_0 \subset M_1 \subset \dots \subset M_n = M$$

such that $M_i = M_{i-1}(t_i)$, $1 \le i \le n$, where either

1. $\partial(t_i) \in M_{i-1}$ (integral)
2. $t_i \neq 0$ and $\partial(t_i) t_i^{-1} \in M_{i-1}$ (exponential of integral)

3. t_i is algebraic $/ \mathcal{F}_{i-1}$.

We will now establish an analog of (+) for Liouville extensions.

A linear algebraic group G is called solvable if it is solvable as an abstract group, i.e., if the derived series, defined via

$$\mathcal{D}^0 G = G \;, \quad \mathcal{D}^{i+1} G := [\mathcal{D}^i G, \mathcal{D}^i G]$$

terminates in the trivial group $\{e\} \leq G$.

Prop 8.2. A linear algebraic group G is solvable iff there exists a chain

$$G = G_0 \geq G_1 \geq \ldots \geq G_n = \{e\}$$

of closed subgroups of G such that, for evey $1 \leq i \leq n$,

1. $G_i \trianglelefteq G_{i-1}$ normal
2. G_{i-1}/G_i abelian.

Proof: HW.

Examples 1.3.

(1) Let $\mathcal{U} := \mathcal{U}(n,k) \leq GL(n,k)$ the subgroup of <u>unipotent matrices</u>, i.e., upper triangular matrices with diagonal entries 1:

$$\mathcal{U} = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \leq GL(n,k) .$$

We denote by $\mathcal{N}_r \subseteq k^{n\times n}$ the set of matrices $X \in k^{n\times n}$ satisfying, for $j-i \leq r$, $X_{ij} = 0$.

We have

$$\mathcal{U} = I_n + \mathcal{N}_1 = \{ I_n + X \mid X \in \mathcal{N}_1 \}$$

and define

$$\mathcal{U}_r := I_n + \mathcal{N}_r \subseteq \mathcal{U}$$

Then, since $\mathcal{N}_r \cdot \mathcal{N}_e \subseteq \mathcal{N}_{r+e}$, and

$$(I+X)^{-1} = I - X + X^2 - \dots + (-1)^{n-1} X^{n-1}$$

(since $X^n = 0$), we conclude (!)

$$[\mathcal{U}_r, \mathcal{U}_e] \subseteq \mathcal{U}_{r+e}$$

But $\mathcal{U}_n = \{I_n\}$ so that $\mathcal{U}$ is solvable.

(2) Let

$$\mathcal{B} := \mathcal{B}(n,K) \leq GL(n,K) \text{ the subgroup}$$

of upper triangular matrices, i.e., with

$$\mathcal{D} = \{\text{invertible diagonal matrices}\} \leq GL(n,K)$$

we have

$$\mathcal{B} = \{D + X \mid D \in \mathcal{D},\ X \in \mathcal{N}_1\}$$
$$= \{D \cdot U \mid D \in \mathcal{D},\ U \in \mathcal{U}\}.$$

We further have

$$D^{-1} U D \in \mathcal{U} \qquad (D \in \mathcal{D},\ U \in \mathcal{U}).$$

This implies(!) $[\mathcal{B},\mathcal{B}] \leq \mathcal{U}$. In particular, by (1), we deduce that $\mathcal{B}$ solvable.

We will use (without proof) the following "converse" of Example 8.3(2):

Theorem 8.4 ("Lie-Kolchin"). Let G be a closed connected solvable subgroup of $GL(n,K)$. Then there exists $T \in GL(n,K)$, such that

$$T^{-1} G T \leq \mathcal{B}(n,K)$$

Proof [H]: §17.6

Example 8.5. Let
$$SL(n,K) \leq GL(n,K)$$
be the special linear group. From linear algebra: $SL(n,K)$ is generated by elementary matrices of the form
$$E_{ij}(\lambda) = I_n + R_{ij}(\lambda) \qquad ; \; i \neq j$$
where $R_{ij}(\lambda)$ is matrix with entry $\lambda \in K$ at position (i,j) and 0 else.
Using this fact, it is straightforward to show
$$[SL(n,K), SL(n,K)] = SL(n,K).$$
In particular: neither $SL(n,K)$ nor $GL(n,K)$ are solvable for $n \geq 2$.

Theorem 8.6. Let E/F Picard-Vessiot extension with ∂-Galois group $G = \mathrm{Gal}^{\partial}(E/F)$. Then the following are equivalent:

(1) G° is solvable.

(2) E/F is a Liouville extension.

(3) E is contained in a Liouville extension.

Proof: (1) $\Rightarrow$ (2):

By ∂-Galois correspondence, we have

$$\begin{array}{cl} E & \\ | & \left.\vphantom{\Big|}\right\} G^\circ \\ E^{G^\circ} & \\ | & \left.\vphantom{\Big|}\right\} G/G^\circ \text{ finite} \Rightarrow E^{G^\circ}/F \text{ algebraic} \\ F & \end{array}$$

Thus, we may assume $G = G^\circ$ connected. Let $Y \in GL(n,E)$ generating fundamental matrix, i.e., $E = F(Y_{i,j})$. Then $\sigma(Y) = Y \cdot C_\sigma$ yields an embedding $G \hookrightarrow GL(n,K)$, $\sigma \longmapsto C_\sigma$.

By Lie-Kolchin Theorem, we replace

$$Y \quad \text{by} \quad Y \cdot T \quad ; \quad T \in GL(n,K)$$

$$\rightsquigarrow \quad G \hookrightarrow \underbrace{B(n,K)}_{\text{upper triangular matrices.}} \subseteq GL(n,K).$$

From

$$\sigma(Y) = Y \cdot C_\sigma \qquad \text{with } C_\sigma \in B(n,K)$$

we have

$$\sigma(Y_{ij}) = Y_{i1}(C_\sigma)_{1j} + \dots + Y_{ij}(C_\sigma)_{jj}$$

so that, by Lemma 8.6 below, for every $\underline{1 \leq i \leq n}$, the extension

$$F(Y_{ij} \mid 1 \leq j \leq n) / F$$

is Liouville. But then, of course,

$$E = F(Y_{ij} \mid 1 \leq i,j \leq n) / F$$

is also Liouville. $\Rightarrow$ (2) ✓.

(2) $\Rightarrow$ (3) ✓

(3) $\Rightarrow$ (1):

Suppose $M = F(t_1, \dots, t_m) / F$ Liouvillian with $E \subseteq M$. We will show that G° is solvable by induction on m. The extension $E(t_1) / F(t_1)$ is Picard-Vessiot, generated by the same fundamental matrix $Y \in GL(n, E)$ such that $E = F(Y_{ij})$.

With respect to this fundamental matrix, we obtain

$$\begin{tikzcd}
 & GL(n,K) & \\
Gal^\partial(E/F) \arrow[ur, bend right, "\text{closed}"] \arrow[d, phantom, "\parallel"] \arrow[rr, phantom, "\cong", near start] & & Gal^\partial(E(t_1)/F(t_1)) \arrow[ul, bend left, "\text{closed}"'] \arrow[ll, dashed, "\text{closed}"] \arrow[d, phantom, "\parallel"] \\
G & & H
\end{tikzcd}$$

Then

$$E^H = E(t_1)^H \cap E = F(t_1) \cap E$$

so that

$$H \cong Gal^\partial(E / F(t_1) \cap E).$$

By induction hypothesis, we have that

H° is solvable.

If $F(t_1) \cap E = F$ then $H = G \Rightarrow H^\circ = G^\circ$ ✓.

So suppose $F(t_1) \cap E \supsetneq F$:

$$G\left\{\begin{array}{l} \left.\begin{array}{c} E \\ | \\ F(t_1)\cap E \end{array}\right\} H \\ \quad\;\; | \\ \quad\;\; F \end{array}\right. \qquad (*)$$

We distinguish 3 cases:

(I) t_1 is algebraic $/F$. Then $F(t_1) \cap E / F$ is algebraic so that (!) $F(t_1) \cap E \subseteq E^{G^\circ}$.

Thus (!): $H^\circ = G^\circ$ ✓

(II) t_1 / F transcendental and $\partial(t_1) = f \in F \setminus \{0\}$.

Then

$$\mathrm{Gal}^\partial(F(t_1)/F) \cong \mathbb{G}_a \qquad (\text{Example from previous lecture}).$$

The only proper Zariski-closed subgroup of $\mathbb{G}_a$ is (!) the trivial group $\{e\}$. Thus, we have

$F(t_1) \cap E = F(t_1) \Rightarrow F(t_1) \subseteq E$.

Therefore, we have (*) :

$$G\left\{\begin{array}{c} E \\ | \\ F(t_1) \\ | \\ F \end{array}\right. \begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} H \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} G/H \cong \mathbb{G}_a \end{array}$$

We deduce, by Lemma 4.7 below, that G^0 is solvable.

(III) t_1/F transcendental and $\partial(t_1) = a t_1$; $a \in F \setminus \{0\}$.

Then

$$\mathrm{Gal}^{\partial}(F(t_1)/F) \cong \mathbb{G}_m .$$

The only proper Z-closed subgroups of $\mathbb{G}_m$ are are the groups $\mu_n(K)$ of roots of unity (!).

We now proceed as in (II) using that

$$\mathrm{Gal}^{\partial}(F(t_1) \cap E/F) \cong \mathbb{G}_m$$

since

$$\mathbb{G}_m\left\{\begin{array}{c} F(t_1) \\ | \\ F(t_1)\cap E \\ | \\ F \end{array}\right. \begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} \mu_n(K) \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \mathbb{G}_m \end{array} \qquad \begin{array}{c} z^n \leftarrow\!\shortmid z \\ \mathbb{G}_m \cong \mathbb{G}_m/\mu_n(K) \end{array}$$

$\xrightarrow{\text{Lemma 4.7}}$ G^0 solvable $\square$

Lemma 8.6. Let E/F Picard-Vessiot extension and suppose that we are given

$$y_1, \dots, y_n \in E$$

such that, for every $\sigma \in \mathrm{Gal}^{\partial}(E/F)$, there exists $C_\sigma \in B(n,K)$ such that

$$\sigma(y_1, \dots, y_n) = (y_1, \dots, y_n) \cdot \underbrace{C_\sigma}_{= C} . \quad (+)$$

Then the extension

$$F(y_1, \dots, y_n)/F$$

is Liouville.

Proof: We argue by induction on n. We may suppose $y_1 \neq 0$. Then we note that, for fixed σ:

$$\sigma(y_1) = y_1 c_{11}$$

so that

$$\frac{\partial(y_1)}{y_1} \in E^G = F .$$

Thus $F(y_1)/F$ Liouvillian. Now for $i \geq 2$, we have

$$\sigma(y_i) = y_1 c_{1i} + \dots + y_i c_{ii} \qquad (*).$$

We divide $(*)$ by $\sigma(y_1)$ $(= y_1 \cdot c_{11})$ and apply ∂ to obtain

$$\partial\left(\sigma\left(\frac{y_i}{y_1}\right)\right) = \cancel{\partial\left(\frac{c_{1i}}{c_{11}}\right)} \; \partial\left(\sigma\left(\frac{y_2}{y_1}\right)\right)\frac{c_{2i}}{c_{11}} + \dots + \partial\left(\sigma\left(\frac{y_i}{y_1}\right)\right)\frac{c_{ii}}{c_{11}}.$$

Thus, the elements

$$\partial\left(\frac{y_2}{y_1}\right), \dots, \partial\left(\frac{y_n}{y_1}\right)$$

satisfy the hypothesis $(+)$ $\Rightarrow$ by induction hypothesis:

$$F\left(\partial\left(\frac{y_2}{y_1}\right), \dots, \partial\left(\frac{y_n}{y_1}\right)\right) / F$$

is Liouville.

Thus

$$F(y_1, \dots, y_n) / F$$

Liouville. $\square$

Lemma 17. Let G linear algebraic group and $H \trianglelefteq G$ closed normal subgroup. Assume that

1. H° is solvable.
2. G/H is abelian.

Then G° is solvable.