Theorem 7.3 has immediate relevance for ∂-Galois theory in the form of the following corollary which is a ∂-analog of the fact from ordinary Galois theory saying that every finite group is isomorphic to the Galois group of some Galois extension.

Corollary 7.4 Let G be a linear algebraic group $/K$ with $K = \overline{K}$ and $\mathrm{char}(K) = 0$. Then there exists a Picard–Vessiot extension E/F such that
$$G \cong \mathrm{Gal}^{\partial}(E/F).$$

Proof: By Theorem 7.3, we may assume that $G \leq GL(n,K)$ Zariski-closed subgroup. By Exercise Sheet 4.4, there exists a Picard–Vessiot extension E/F such that $\mathrm{Gal}^{\partial}(E/F) \cong GL(n,K)$. Therefore, by the ∂-Galois correspondence, we have
$$\mathrm{Gal}^{\partial}(E/E^{G}) \cong G.$$
$\square$

We now discuss quotients of linear algebraic groups. A proper discussion of this topic would take too long. For this reason, we cite the relevant results which can be found in Chapter 12 of

[H]: James E. Humphreys - Linear algebraic groups.

Definition 7.5. Let G be a linear algebraic group and $H \trianglelefteq G$ Zariski-closed normal subgroup. A quotient of G by H is a linear algebraic group G/H equipped with a homomorphism $\pi : G \to G/H$ with $\mathrm{kernel}(\pi) = H$ such that, for $\varphi : G \to G'$ homomorphism with $H \subseteq \ker(\varphi)$, there exists a unique hom $\overline{\varphi} : G/H \to G'$ such that

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & G' \arrow[d, phantom, "\equiv"] \\ & G/H \arrow[u, "\overline{\varphi}"'] \end{tikzcd}$$

commutes.

We will need the following results from [H]:

H1. Under the hypothesis of 7.5, a quotient exists and is unique up to isomorphism. The underlying abstract group structure on G/H is the quotient of the abstract groups underlying G and H.

H2. The pullback homomorphism π^* factors as

$$
\begin{tikzcd}
\mathcal{O}(G/H) \arrow[r, "\cong"] \arrow[rr, bend right, "\pi^*"'] & \mathcal{O}(G)^H \arrow[r, hook] & \mathcal{O}(G)
\end{tikzcd}
$$

where H acts on G (and hence on $\mathcal{O}(G)$) via right multiplication.

H3. For a commutative ring R, we denote the localization

$$Qt(R) := S^{-1}R$$

at $S = \{\text{non-zero divisors}\}$, called the total ring of fractions of R. (If R is integral domain

then $Qt(R) = Quot(R)$ field of fractions).
Under the hypothesis of 7.5, the natural map

$$Qt(\tilde{F} \otimes_K \mathcal{O}(G)^H) \longrightarrow Qt(\tilde{F} \otimes_K \mathcal{O}(G))^H$$

is an isomorphism ($\tilde{F}/K$ some field extension).

Example 7.6. On the last problem set, we have seen that

$$GL(n,K)/H \cong PGL(n,K)$$

($H = \{\text{scalar matrices}\} \cong K^*$) admits the structure of a linear algebraic group together with the homomorphism

$$\pi : GL(n,K) \twoheadrightarrow PGL(n,K)$$

from 7.5.

Equipped with these results, we will now refine our understanding of the ∂-Galois correspondence with respect normal subgroups.

Theorem 7.6. Let E/F Picard-Vessiot extension with ∂-Galois group $G = \mathrm{Gal}^{\partial}(E/F)$. Let $H \leq G$ Z-closed subgroup. Then

$$H \trianglelefteq G \text{ normal} \iff E^H \text{ } G\text{-stable}.$$

If $H \trianglelefteq G$ normal, then we further have

(1) E^H/F is a Picard-Vessiot extension.

(2) The restriction map

$$G \longrightarrow \mathrm{Gal}^{\partial}(E^H/F)$$

induces an isomorphism $G/H \cong \mathrm{Gal}^{\partial}(E^H/F)$.

Proof: "$\Rightarrow$" ✓. "$\Leftarrow$": Suppose that E^H is G-stable then, by definition, H is the kernel of the restriction homomorphism

$$G \longrightarrow \mathrm{Aut}^{\partial}(E^H/F)$$

and thus $H \trianglelefteq G$.

We show (1). Suppose E is the field of fractions of a Picard-Vessiot ring R for $A \in F^{n\times n}$.

Then, the affine F-scheme X corresponding to R is, by the Torsor theorem, a G_F-torsor.

By Hilbert's Nullstellensatz (Version II), there exists a finite field extension $\tilde{F}/F$ such that

$$X(\tilde{F}) \neq \emptyset .$$

Thus, by Cor. 5.10, we obtain an isomorphism of affine $\tilde{F}$-schemes

$$X_{\tilde{F}} \cong G_{\tilde{F}} \qquad (*) .$$

Wlog, we may assume $\tilde{F}/F$ is a finite Galois extension ($\mathrm{char}(F)=0$).

The isomorphism $(*)$ corresponds to an isomorphism

$$\tilde{F} \otimes_F R \cong \tilde{F} \otimes_K \mathcal{O}(G) \qquad (*)^{op}$$

which is G-equivariant for the G-action on the R (resp. $\mathcal{O}(G)$) - tensor component.

By passing to H-invariants, we thus obtain an isomorphism of $\tilde{F}$-algebras

$$\tilde{F} \otimes_F R^H \cong \tilde{F} \otimes_K \mathcal{O}(G)^H .$$

Since, by H2, the K-algebra

$$\mathcal{O}(G)^H \cong \mathcal{O}(G/H) \text{ is f.g.}$$

is f.g. K-algebra, we have that $\tilde{F} \otimes_K \mathcal{O}(G)^H$ and hence $\tilde{F} \otimes_F R^H$ is a f.g. $\tilde{F}$-algebra.

By Lemma 7.7 below, this implies that R^H is finitely generated as F-algebra (Galois descent).

Let $y_1, \dots, y_n \in R^H$ be generators. Arguing as in the proof of Proposition 7.2 (!), it follows that, for every element $f \in \tilde{F} \otimes_K \mathcal{O}(G)$, the K-vector space $\langle G.f \rangle$ is finite dimensional.

Thus, each $\langle G.y_i \rangle$ is finite-dimensional (and $\subseteq R^H$).

Replacing $\{y_1, \dots, y_n\}$ by a K-basis of the K-vector space $\langle \{G.y_i\} \mid 1 \leq i \leq n \rangle$ we may assume that $y_1, \dots, y_n$ forms the K-basis of a G-stable subspace of R^H.

Thus, the Wronskian matrix

$$Y := \mathrm{Wr}(y_1, \dots, y_n)$$

has nonzero determinat (Prop 2.7) and is thus invertible when considered as a matrix with coefficiets in $\mathrm{Quot}(R^H) \subseteq E$.

Further, the matrix

$$B := \partial(Y)\, Y^{-1}$$

is G-invariant (c.f. Exercise Sheet 3.4), so that B has entries in $E^G = F$.

Finally, we have, by Lemma 7.8 below, that

$$\begin{aligned}
\tilde{F} \otimes_F \mathrm{Gt}(R^H) &\cong \mathrm{Gt}(\tilde{F} \otimes_F R^H) \\
&\cong \mathrm{Gt}(\tilde{F} \otimes_C \mathcal{O}(G)^H) \\
&\cong \mathrm{Gt}(\tilde{F} \otimes_C \mathcal{O}(G))^H \qquad (H3) \\
\overset{7.8}{} &\cong \mathrm{Gt}(\hat{F} \otimes_F R)^H \\
&\cong \hat{F} \otimes \mathrm{Gt}(R)^H .
\end{aligned}$$

Tracing through the isomorphisms, we obtain that the F-lin.:

$$i : \mathrm{Quot}(R^H) = \mathrm{Gt}(R^H) \longrightarrow \mathrm{Gt}(R)^H = E^H$$

becomes an isomorphism after applying $\tilde{F} \otimes_F -$.
Therefore (!) the map i must itself already be an isomorphism (argue with an F-basis of $\mathrm{Quot}(R^H)$).

Thus, we have

$$E^H = \mathrm{Quot}(R^H) = F(Y_{i,j} \mid 1 \le i,j \le n)$$

$\Rightarrow$ E^H/F is Picard-Vessiot extension.

To show (2): We show that the restriction hom

$$\pi : G \longrightarrow \mathrm{Gal}^\partial(E^H/F)$$

is surjective ($\Rightarrow G/H \cong \mathrm{Gal}^\partial(E^H/F)$ since $\ker(\pi) = H$).

So let $\sigma : E^H \to E^H$ ∂-automorphism.

Then we consider

$$\left.\begin{aligned} &\mathrm{inc} : E^H \hookrightarrow E \\ &\text{and} \\ &\mathrm{inc} \circ \sigma : E^H \hookrightarrow E. \end{aligned}\right\} \text{both are PV-extensions for } A \text{ over } E^H.$$

Thus there exists an isomorphism of ∂-fields

$$\tilde{\sigma} : E \longrightarrow E$$

such that

$$\begin{tikzcd} E \arrow[rr, "\tilde{\sigma}"] & & E \\ & E^H \arrow[ul, "inc"] \arrow[ur, "inc \circ \sigma"'] \arrow[u, phantom, "="] & \end{tikzcd} \iff \begin{tikzcd} E \arrow[r, "\tilde{\sigma}"] & E \\ E^H \arrow[u, "inc"] \arrow[r, "\sigma"'] \arrow[ur, phantom, "="] & E^H \arrow[u, "inc"'] \end{tikzcd}$$

Thus $\tilde{\sigma}|E^H = \sigma = \pi(\sigma)$. $\square$.

Lemma 7.7. Let $\tilde{F}/F$ finite Galois extension and let S F-algebra. Then

$$S \text{ f.g. } F\text{-alg} \iff \tilde{F} \otimes_F S \text{ f.g. as } \tilde{F}\text{-alg}.$$

Lemma 7.8. Let $\tilde{F}/F$ finite Galois extension and S F-algebra. Then

$$\mathrm{Gt}(\tilde{F} \otimes_F S) \cong \tilde{F} \otimes_F \mathrm{Gt}(S).$$