Differential Galois Theory

§1. Introduction

Reminder: Classical Galois Theory

We are given an irreducible polynomial

$$f(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in \mathbb{Q}[X].$$

Fundamental Theorem of Algebra

$$\leadsto \quad f(X) = (X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_n)$$

in $\mathbb{C}[X]$, where $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ roots of f.

Definition 1.1. The Galois group $Gal(f) \leq S_n$ is the subgroup consisting of those $\sigma \in S_n$ which satisfy the following condition:

* for every $F(X_1, \dots, X_n) \in \mathbb{Q}[X_1, \dots, X_n]$, the implication

$$F(\alpha_1, \dots, \alpha_n) = 0 \Rightarrow F(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) = 0$$

holds

Remark 1.2. Put in plain terms, Definition 1.1 says that the group $Gal(f)$ consists precisely of those permutations which preserve all $\mathbb{Q}$-algebraic relations among $\alpha_1, \dots, \alpha_n$.

Example 1.3. $f(X) = X^4 - 2$.

The roots are given by

$$\alpha_1 = \sqrt[4]{2}\ ; \quad \alpha_2 = i\sqrt[4]{2}\ ; \quad \alpha_3 = -\sqrt[4]{2}, \quad \alpha_4 = -i\sqrt[4]{2}.$$

Each root satisfies the equation

$$\alpha_i^4 = 2.$$

Further, due to

$$(X-\alpha_1)(X-\alpha_2)(X-\alpha_3)(X-\alpha_4) = X^4 - 2$$

we have

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = 0$$

$$\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 = 0$$

$$\alpha_1\alpha_2\alpha_3\alpha_4 = -2$$

There are additional "hidden" relations:

(I) $(\alpha_1\alpha_2)^2 = -2$

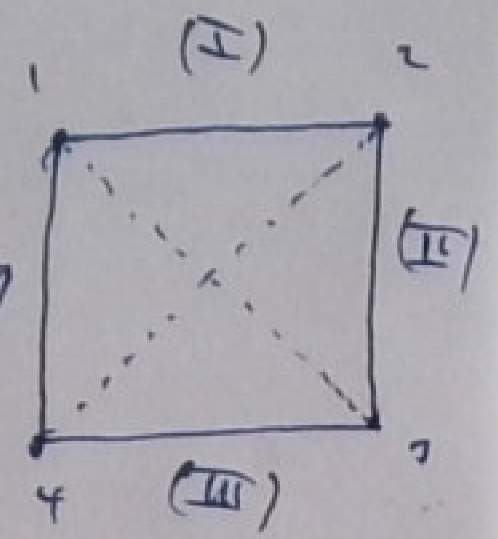
(II) $(\alpha_2\alpha_3)^2 = -2$

(III) $(\alpha_3\alpha_4)^2 = -2$

(IV) $(\alpha_4\alpha_1)^2 = -2$

(V) $(\alpha_1\alpha_3)^2 = 2$

(VI) $(\alpha_2\alpha_4)^2 = 2$

~ 

$\rightsquigarrow$ the permutations that preserve (I) - (VI) form the dihedral group

$$D_4 \le S_4$$

Turns out that all other relations among $\{\alpha_i\}$ are consequences of the above relations

$\rightsquigarrow$ $$Gal(f) = D_4.$$

Example 1.4. Consider

$$f(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 \in \mathbb{Q}[X]$$

irreducible polynomial, with

$$\alpha_1, \dots, \alpha_n \in L$$

roots in some splitting field $L/\mathbb{Q}$.

The number

$$D = \prod_{i<j} (\alpha_i - \alpha_j)^2 \in \mathbb{Q}$$

is called the discriminant.

In L, the discriminant has a square root:

$$\delta = \prod_{i<j} (\alpha_i - \alpha_j) \in L.$$

We have:

[The polynomial $Y^2 - D$ has a rational root $\Longleftrightarrow$ $Gal(f) \le A_n$]

Preview: differential Galois theory

Consider

$$\ell = \partial^n + f_{n-1}\partial^{n-1} + \dots + f_0 \overbrace{\partial^0}^{id}$$

differential operator where

- $f_i \in \mathbb{C}(t) := \mathrm{Quot}(\mathbb{C}[t])$
- $\partial = \frac{d}{dt}$

⇝ associated differential equation: $\ell(y) = 0$.

Suppose $t = t_0 \in \mathbb{C}$ is a regular point of ℓ, i.e., the coefficients $\{f_i\}$ do not have any poles at t_0.

Picard–Lindelöf Theorem

$$\Rightarrow \quad Sol(\ell) := \{ y \in \mathbb{C}[[t-t_0]] \mid \ell(y) = 0 \}$$

is an n-dimensional $\mathbb{C}$-vector space.

We pick a basis

$$B = \{y_1, \dots, y_n\}$$

of $Sol(\ell)$.

Definition 1.5. The differential Galois group

$$Gal^{\partial}(\ell) \leq GL(n, \mathbb{C}) = GL(Sol(\ell))$$

is the subgroup of those linear transformations of $Sol(\ell)$ which preserve all algebraic relations among

$$y_1, y_2, \dots, y_n, \partial(y_1), \dots, \partial(y_n), \dots, \partial^{n-1}(y_1), \dots, \partial^{n-1}(y_n)$$

with coefficients in $\mathbb{C}(t)$.

Example 1.6. Consider

$$\ell = \partial - id$$

differential operator with associated $\overset{\text{"differential"}}{\partial}$-equation

$$\ell(y) = 0 \quad \sim \quad y' = y \qquad (*)$$

Choose the regular point $t_0 = 0$, then we have the solution

$$\exp(t) := 1 + t + \frac{t^2}{2} + \dots \in \mathbb{C}[[t]],$$

and

$$Sol(\ell) = \{ \lambda \cdot \exp(t) \mid \lambda \in \mathbb{C} \} \cong \mathbb{C}$$

is a 1-dimensional vector space.

Preview: differential Galois theory

Consider

$$\ell = \partial^n + f_{n-1}\partial^{n-1} + \dots + f_0 \overbrace{\partial^0}^{\mathrm{id}}$$

differential operator where

- $f_i \in \mathbb{C}(t) := \mathrm{Quot}(\mathbb{C}[t])$
- $\partial = \frac{d}{dt}$

⇝ associated differential equation: $\ell(y) = 0$.

Suppose $t = t_0 \in \mathbb{C}$ is a regular point of ℓ, i.e., the coefficients $\{f_i\}$ do not have any poles at t_0.

Picard–Lindelöf Theorem

$$\Rightarrow \quad \mathrm{Sol}(\ell) := \{ y \in \mathbb{C}[[t - t_0]] \mid \ell(y) = 0 \}$$

is an n-dimensional $\mathbb{C}$-vector space.

We pick a basis

$$B = \{y_1, \dots, y_n\}$$

of $\mathrm{Sol}(\ell)$.

Definition 1.5. The differential Galois group

$$\mathrm{Gal}^{\partial}(\ell) \leq GL(n, \mathbb{C}) = GL(\mathrm{Sol}(\ell))$$

is the subgroup of those linear transformations of $\mathrm{Sol}(\ell)$ which preserve all algebraic relations among

$$y_1, y_2, \dots, y_n, \partial(y_1), \dots, \partial(y_n), \dots, \partial^{n-1}(y_1), \dots, \partial^{n-1}(y_n)$$

with coefficients in $\mathbb{C}(t)$.

Example 1.6. Consider

$$\ell = \partial - \mathrm{id}$$

"differential"

differential operator with associated ∂-equation

$$\ell(y) = 0 \quad \sim \quad y' = y \qquad (*)$$

Choose the regular point $t_0 = 0$, then we have the solution

$$\exp(t) := 1 + t + \frac{t^2}{2} + \dots \in \mathbb{C}[[t]],$$

and

$$\mathrm{Sol}(\ell) = \{ \lambda \cdot \exp(t) \mid \lambda \in \mathbb{C} \} \cong \mathbb{C}$$

is a 1-dimensional vector space.

the ∂-Galois group
$$\mathrm{Gal}^{\partial}(\ell) \leq GL(1,\mathbb{C}) = \mathbb{C}^* .$$
Since $\exp(t)$ is transcendental over $\mathbb{C}(t)$ (!), it does not satisfy any nontrivial algebraic relation/$\mathbb{C}(t)$.
$$\Rightarrow \quad \mathrm{Gal}^{\partial}(\ell) = \mathbb{C}^* \quad \text{multiplicative group}.$$

Example 1.7. Would like to consider the equation
$$y' = \frac{1}{t} . \quad (*)$$
Issue: not homogeneous.

Trick: differentiate again $\rightsquigarrow$ $y'' = -\frac{1}{t^2} \overset{(*)}{=} -\frac{1}{t} y'$ $(**)$.

Equation $(**)$ is associated to the operator
$$\ell = \partial^2 + \frac{1}{t}\partial .$$
Choose $t_0 = 1$ regular point. Solutions to $(**)$:
$$\left.\begin{aligned} y_1 &= 1 \\ y_2 &= \ln t = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (t-1)^k \end{aligned}\right\} \in \mathbb{C}[[t-1]] .$$

Then
$$\mathrm{Sol}(\ell) = \{ \lambda_1 + \lambda_2 \ln t \mid \lambda_1, \lambda_2 \in \mathbb{C} \} \cong \mathbb{C}^2$$
so
$$\mathrm{Gal}^{\partial}(\ell) \leq GL(2,\mathbb{C}) .$$
Algebraic relations among:
$$\underset{\substack{\parallel \\ 1}}{y_1},\ \underset{\substack{\parallel \\ \ln t}}{y_2},\ \underset{\substack{\parallel \\ 0}}{\partial(y_1)},\ \underset{\substack{\parallel \\ \frac{1}{t}}}{\partial(y_2)}$$
$$\rightsquigarrow \quad \left.\begin{aligned} &(\mathrm{I}) \quad y_1 = 1 \\ &(\mathrm{II}) \quad \partial(y_2) = \frac{1}{t} . \end{aligned}\right\} \text{these relations imply all other relations (!)}$$
These relations imply that transformations in $\mathrm{Gal}^{\partial}(\ell)$ need to be of the form
$$\underbrace{\begin{aligned} y_1 &\longmapsto y_1 \\ y_2 &\longmapsto y_2 + c \end{aligned}} \qquad \text{with } c \in \mathbb{C} .$$
$$\mathrm{Gal}^{\partial}(\ell) = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \,\middle|\, c \in \mathbb{C} \right\} \leq GL(2,\mathbb{C})$$
$\cong$

so $\mathrm{Gal}^{\partial}(\ell) \cong (\mathbb{C}, +)$ additive group.

Example 1.8. Consider

$$\ell = \partial^n + f_{n-1}\partial^{n-1} + \dots + f_0 \cdot \mathrm{id} \in \mathbb{C}(t)[\partial]$$

differential operator. Let

$$y_1, \dots, y_n \in \mathbb{C}[[t-t_0]]$$

basis for $\mathrm{Sol}(\ell)$ with $t_0 \in \mathbb{C}$ regular point.

Form the Wronskian:

$$w = \det \begin{pmatrix} y_1 & \cdots & y_n \\ \partial(y_1) & \cdots & \partial(y_n) \\ \vdots & & \vdots \\ \partial^{n-1}(y_1) & \cdots & \partial^{n-1}(y_n) \end{pmatrix} \in \mathbb{C}[[t-t_0]]$$

The ∂-Galoisgroup $\mathrm{Gal}^{\partial}(\ell)$ acts via ∂-ring homomorphisms on

$$\mathbb{C}(t)[\{y_i\}, \{\partial(y_i)\}, \dots, \{\partial^{n-1}(y_i)\}]$$

and we have for $C \in \mathrm{Gal}^{\partial}(\ell) \leq GL(n, \mathbb{C})$:

$$C.w = \det(C).w$$

Further, the Wronskian satisfies (!) the ∂-equation

$$\partial(w) = -f_{n-1} \cdot w \qquad (*)$$

and we have

The ∂-equation

$$\frac{dz}{dt} = -f_{n-1} \cdot z$$

has a solution in $\mathbb{C}(t)$ $\iff$ $\mathrm{Gal}^{\partial}(\ell) \leq SL(n, \mathbb{C})$

First goal of the course:

Put the foundations of ∂-Galois theory on a purely algebraic footing

"Picard-Vessiot Theory"