Exercise sheet 9

Problem 1. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of small categories.

1. Show that the nerve $N(f) : N(\mathcal{C}) \to N(\mathcal{D})$ has the right lifting property with respect to all inner horn inclusions $\Lambda^n_i \subset \Delta^n$.

2. Suppose that $N(f)$ has the right lifting property with respect to the horn inclusions $\Lambda^n_0 \subset \Delta^n$. Show that the fibers of f are groupoids and that the map $N(f)$ is a Kan fibration.

Problem 2. Recall from exercise 4.4 that there is a functor $D : \Delta \to 2\mathbf{Cat}$, which defines the *2-nerve* $N_2 : 2\mathbf{Cat} \to \mathbf{Set}_\Delta$ via the Yoneda embedding. Recall from the lecture the adjunction

$$\mathfrak{C} : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Cat}_{\mathbf{Set}_\Delta} : N_\Delta$$

defined via the functor $\mathfrak{C} : \Delta \to \mathbf{Cat}_{\mathbf{Set}_\Delta}$.

1. Show that, given a 2-category $\mathcal{C}$, by applying the nerve to the various categories $\mathcal{C}(x, y)$, we obtain a simplicial category. Further show that this construction yields a functor $N_M : 2\mathbf{Cat} \to \mathbf{Cat}_{\mathbf{Set}_\Delta}$.

2. Show that $N_M \circ D = \mathfrak{C}$. Conclude that $N_\Delta \circ N_M = N_2$.

3. Let $\mathcal{C} \in 2\mathbf{Cat}$. Show that $N_2(\mathcal{C})$ is an ∞-category if and only if all categories $\mathbb{C}(x, y)$ are groupoids.

Problem 3. Define $[-1] := \varnothing$, and for $n, m \in \{-1, 0, 1, \ldots\}$, define the *ordinal sum* $[n] \oplus [m]$ of $[n]$ and $[m]$ to be the totally ordered set

$$\{0 < 1 \cdots < n < 0' < 1' \cdots < m'\} \cong [n + m + 1].$$

Let $X, Y \in \mathbf{Set}_\Delta$. Define the *join* of X and Y to be the simplicial set $X \star Y$ whose n-simplices are given by

$$(X \star Y)_n = \coprod_{[i]\oplus[j]\cong[n]} X_i \times X_j,$$

where we set $X_{-1} = *$.

1. Show that $X \star Y$ is a simplicial set. Note that there are canonical monomorphisms of simplicial sets $X \hookrightarrow X \star Y$ and $Y \hookrightarrow X \star Y$.

2. Show that $\Delta^n \star \Delta^m \cong \Delta^{n+m+1}$

3. Show that if X and Y are ∞-categories, then so is $X \star Y$.

4. Let $\mathcal{C} \in \mathbf{Cat}$, and let $x \in \mathcal{C}$ be an object. Show that the nerve of the overcategory $N(\mathcal{C}_{/x})$ can be identified with the simplicial set $N(\mathcal{C})_{/x}$ whose n-simplices are morphisms of simplicial sets $\Delta^n \star \Delta^0 \to N(\mathcal{C})$ such that the composite $\Delta^0 \to \Delta^n \star \Delta^0 \to N(\mathcal{C})$ is x.

Problem 4. Define the sets of morphisms

$$\mathcal{L} := \{\Lambda_i^n \hookrightarrow \Delta^n \mid 0 \leq i < n\}$$
$$\mathcal{R} := \{\Lambda_i^n \hookrightarrow \Delta^n \mid 0 < i \leq n\}.$$

Call a morphism $f : X \to Y$ of simplicial sets *left anodyne* (resp. *right anodyne*) if it is in $\overline{\mathcal{L}}$ (resp. $\overline{\mathcal{R}}$). Let $g : A \to B$ be a monomorphism in $\mathbf{Set}_\Delta$. Show that if $f : A' \to B'$ is left anodyne, then $f \wedge g$ is left anodyne as well. (Hint: See the proof of the analogous fact for anodyne morphisms.) Conclude that if f is right anodyne, so is $f \wedge g$.