Exercise Sheet 8

Problem 1. Let I be the category with two objects 0 and 1 and a unique isomorphism between any two objects. Define the set $\mathfrak{C}$ of morphisms of **Cat** to be those which are injective on objects. Define the set $\mathfrak{F}$ of morphisms of **Cat** to be the class of morphisms which have the right lifting property with respect to the functor $\{0\} \to I$, i.e. the isofibrations.

1. Let $F : C \to D$ be a functor.
   (a) Consider the category $\mathcal{L}$ whose objects are tuples (c, d, ϕ), where $c \in \mathcal{C}$, $d \in \mathcal{D}$, and $\phi : F(c) \overset{\cong}{\to} d$ in $\mathcal{D}$, and whose morphisms are given by
   $$\mathcal{L}\left((c, d, \phi),\ (a, b, \psi)\right) := \mathcal{C}(c, a).$$
   Show that F can be factored as $\mathcal{C} \overset{G}{\to} \mathcal{L} \overset{H}{\to} \mathcal{D}$, such that G is an equivalence of categories, $G \in \mathfrak{C}$, and $H \in \mathfrak{F}$.
   (b) Let $\mathcal{R}$ be the category whose objects are given by $\mathrm{Ob}(\mathcal{C}) \coprod \mathrm{Ob}(\mathcal{D})$, with morphisms characterized by (for $d_1, d_2 \in \mathcal{D}$, and $c_1, c_2 \in \mathcal{C}$)
   $$\begin{array}{rcl} \mathcal{R}\,(c_1, c_2) & := & \mathcal{D}(F(c_1), F(c_2)) \\ \mathcal{R}\,(d_1, d_2) & := & \mathcal{D}(d_1, d_2) \\ \mathcal{R}\,(c_1, d_1) & := & \mathcal{D}(F(c_1), d_1)) \\ \mathcal{R}\,(d_1, c_1) & := & \mathcal{D}(d_1, F(c_1)). \end{array}$$
   Show that F can be factored as $\mathcal{C} \overset{G}{\to} \mathcal{R} \overset{H}{\to} \mathcal{D}$, such that H is an equivalence of categories, $G \in \mathfrak{C}$, and $H \in \mathfrak{F}$.

2. Let
   $$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{A} \\ {\scriptstyle F}\downarrow & & {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{B} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{B} \end{array}$$
   be a diagram of categories such that the horizontal composites are identities. Show that
   (a) If G is an equivalence of categories, so is F (Hint: consider the adjoint to F.)
   (b) If G is in $\mathfrak{C}$, so is F (Hint: consider the forgetful functor to **Set**.)
   (c) If G is in $\mathfrak{F}$, so is F.

3. Given a diagram of categories
   $$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$
   such that $\iota \in \mathfrak{C}$ and $P \in \mathfrak{F}$, show that
   (a) If P is an equivalence of categories, then there exists a lift $\ell : \mathcal{C} \to \mathcal{B}$. (Hint: first show that P must be surjective on objects.)
   (b) If ι is an equivalence of categories, then there exists a lift $\ell : \mathcal{C} \to \mathcal{B}$.

4. Deduce that there is a model structure on the category of small categories such that the weak equivalences are the equivalences of categories, the fibrations are the morphisms in $\mathfrak{F}$, and the cofibrations are the morphisms in $\mathfrak{C}$.

Problem 2. Let $(\mathbf{Cat}, \mathfrak{W}, \mathfrak{C}, \mathfrak{F})$ be the model category from problem 1.

1. Identify the fibrant and cofibrant objects in $(\mathbf{Cat}, \mathfrak{W}, \mathfrak{C}, \mathfrak{F})$.
2. Show that $\mathcal{C} \amalg \mathcal{C} \to \mathcal{C} \times I \to \mathcal{C}$ is a cylinder object on $\mathcal{C}$ in $(\mathbf{Cat}, \mathfrak{W}, \mathfrak{C}, \mathfrak{F})$.
3. Show that a left homotopy between $F, G : \mathcal{C} \to \mathcal{D}$ in $(\mathbf{Cat}, \mathfrak{W}, \mathfrak{C}, \mathfrak{F})$ is the same thing as a natural isomorphism between F and G.