Exercise sheet 5

Problem 1. (1) Let K be a small field. Show that $V \in \mathbf{Vect}_K$ is a compact object if and only if V is finite dimensional.

(2) Let $\mathcal{K} \subset \mathcal{P}(0, 1, \ldots, n)$. Show that $\Delta^{\mathcal{K}}$ is a compact object in $\mathbf{Set}_\Delta$.

(3) (Bonus) Characterize the compact objects in $\mathbf{Set}_\Delta$.

Problem 2. Call a functor $F : I \to J$ between small categories *cofinal* if, for every $j \in J$, the geometric realization of the nerve of the slice category $|N(j/I)|$ (often written as $|j/I|$) is non-empty and connected.

(1) Let $\mathcal{C}$ be a category with all small colimits and let $G : J \to \mathcal{C}$ be a diagram. Show that if F is cofinal,

$$\operatorname{colim} G \circ F \cong \operatorname{colim} G.$$

(2) Show that right adjoints are cofinal. Conclude that the inclusion of a terminal object is cofinal.

Problem 3. (1) Consider the set

$$\mathcal{M} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$$

of morphisms in $\mathbf{Set}_\Delta$. Show that the saturated hull $\overline{\mathcal{M}}$ is the set of monomorphisms in $\mathbf{Set}_\Delta$.

(2) Let $i : A \to A'$ be a fixed morphism in $\mathbf{Set}_\Delta$. Consider the set $\mathcal{S}$ of morphisms in $\mathbf{Set}_\Delta$ consisting of those morphisms $B \to B'$ such that the morphism

$$A' \times B \coprod_{A \times B} A \times B' \longrightarrow A' \times B'$$

is anodyne. Show that $\mathcal{S}$ is saturated.

(3) Deduce that the sets

$$\mathcal{M}_2 = \{\Delta^n \times \{e\} \coprod_{\partial\Delta^n \times \{e\}} \partial\Delta^n \times \Delta^1 \longrightarrow \Delta^n \times \Delta^1 \mid n \geq 0, e = 0, 1\}$$

and

$$\mathcal{M}_3 = \{B \times \{e\} \coprod_{A \times \{e\}} A \times \Delta^1 \longrightarrow B \times \Delta^1 \mid A \hookrightarrow B \text{ monomorphism}, e = 0, 1\}$$

have the same saturated hull (given by the set of anodyne morphisms).

Problem 4. Let $\mathcal{S}$ denote the set of all surjections in $\mathbf{Grp}$. Show that

(1) A homomorphism of groups $h : G \to H$ is in $\mathcal{S}$ if and only if it has the right lifting property with respect to $1 \to \mathbb{Z}$.

(2) A group G is a free group if and only if the morphism $1 \to G$ has the left lifting property with respect to all morphisms in $\mathcal{S}$.

Problem 5. Denote by **Ch** the category of chain complexes of small abelian groups in non-negative degree. We define

$$\mathbb{Z}[n]_k := \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{else} \end{cases}$$

(in particular, if $k < 0$, then $\mathbb{Z}[n]$ is the trivial complex) and

$$\mathbb{Z}[n, n-1] := \begin{cases} \mathbb{Z} & k = n, n-1 \\ 0 & \text{else.} \end{cases}$$

All of the differentials of $\mathbb{Z}[n]$ are 0, and the only non-zero differential of $\mathbb{Z}[n, n-1]$ is the identity map $\mathbb{Z} \to \mathbb{Z}$. Define two sets of morphisms in **Ch**

$$\begin{aligned} \mathcal{A} &:= \{0 \to \mathbb{Z}[n, n-1] \mid n \geq 1\} \\ \mathcal{B} &:= \{\mathbb{Z}[n-1] \to \mathbb{Z}[n, n-1] \mid n \geq 0\}. \end{aligned}$$

(1) Show that $f : V_\bullet \to W_\bullet$ has the right lifting property with respect to all morphisms in $\mathcal{A}$ if and only if, for every $n \in \mathbb{N}$, the map $f_n : V_n \to W_n$ is surjective.

(2) Show that $f : V_\bullet \to W_\bullet$ has the right lifting property with respect to all morphisms in $\mathcal{B}$ if and only if f is satisfies the following conditions:

   (a) for every $n \in \mathbb{N}$, the map $f_n : V_n \to W_n$ is surjective,

   (b) for every $n \in \mathbb{N}$, the induced map $f_n : H_n(V_\bullet) \to H_n(W_\bullet)$ is an isomorphism.