Introduction to to higher category theory — Winter Semester 18/19
Prof. Dr. Tobias Dyckerhoff — Due: 13. November

Exercise sheet 4

Problem 1. Let I be a small category, let $\mathcal{C}$ be a locally small category, and let $\mathcal{C}$ admit all small colimits. Let $F : I \to \mathcal{C}$ be a functor, and define

$$G : \mathcal{C} \to \mathbf{Set}_I; \quad c \mapsto \mathcal{C}(F(-), c).$$

1. Show that the left Kan extension $Y_!F$ of F along the Yoneda embedding is left adjoint to G. Give an explicit formula for $Y_!F$. (Hint: Compare with the proof of the $|-|$-Sing adjunction from class.)

2. Show that the nerve functor $N : \mathbf{Cat} \to \mathbf{Set}_\Delta$ has a left adjoint τ_1.

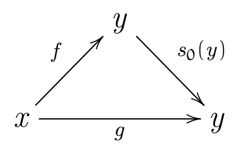
3. Consider the functor $E : \Delta \to \mathbf{Grpd}$ which sends $[n]$ to the groupoid $E([n])$ with objects $0, 1, \dots n$ and a unique isomorphism between every pair of objects. Show that the functor

$$M : \mathbf{Grpd} \to \mathbf{Set}_\Delta; \quad \mathcal{C} \mapsto \mathbf{Grpd}\,(E(-), \mathcal{C})$$

is naturally isomorphic to the restriction of N to $\mathbf{Grpd}$. Show that M admits a left adjoint. Is there a relation between the left adjoint of M and τ_1?

Problem 2. Denote by $\mathbf{qCat}$ the full subcategory of $\mathbf{Set}_\Delta$ on the simplicial sets which have all *inner horn fillers* (i.e. all horn fillers for $\Lambda^n_i \to \Delta^n$ where $n \geq 2$ and $0 < i < n$). Denote by $\mathbf{Kan}$ the full subcategory of $\mathbf{Set}_\Delta$ on the Kan complexes.

1. For $X \in \mathbf{qCat}$, define a relation on the edges of X as follows. Two edges $f, g \in X_1$ with $d_0f(f) = d_0(g) = y$ and $d_1(f) = d_1(g) = x$ are said to be homotopic ($f \sim g$) if there is a 2-simplex $\sigma \in X_2$ such that $d_0(\sigma) = s_0(y)$, $d_1(\sigma) = g$, and $d_2(\sigma) = f$. Pictorially



is the 2-simplex σ. Show that $\sim$ is an equivalence relation.

2. To each $X \in \mathbf{qCat}$ associate the category $\gamma(X)$ whose objects are the 0-simplices of X, and whose morphisms from x to y are equivalence classes of 1-simplicies under the equivalence relation from part 1. Show that this construction yields a well-defined functor $\gamma : \mathbf{qCat} \to \mathbf{Cat}$.

3. Show that γ is left adjoint to the nerve functor $N : \mathbf{Cat} \to \mathbf{qCat}$. In particular, note that for $X \in \mathbf{qCat}$ there is a natural isomorphism $\gamma(X) \cong \tau_1(X)$.

4. Prove that a simplicial set X is the nerve of a category if and only if has *unique* fillers for all inner horns.

5. Prove that if $X \in \mathbf{Kan}$, $\gamma(X)$ is a groupoid.

Problem 3. For each standard simplex $\Delta^n \in \mathbf{Set}_\Delta$, define $P\Delta^n$ to be the poset of non-degenerate simplices in Δ^n.

1. Show that the assignment $[n] \mapsto P\Delta^n$ defines a functor $P : \Delta \to \mathbf{Cat}$.
2. Define $\mathrm{sd} : \Delta \to \mathbf{Set}_\Delta$ by $[n] \mapsto N(P\Delta^n)$. Use problem 1 to define an adjunction

$$\mathrm{sd} : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Set}_\Delta : \mathrm{Ex}\,.$$

   And give formulas for $\mathrm{Ex}(X)$ and $\mathrm{sd}(X)$ for any $X \in \mathbf{Set}_\Delta$.
3. Show that, for any $X \in \mathbf{Set}_\Delta$, there is homeomorphism of topological spaces

$$|X| \to |\,\mathrm{sd}\, X|.$$

4. Define a map of posets $P\Delta^n \to [n]$ given by sending a simplex $\{i_1, i_2, \ldots, i_k\}$ to $i_k \in [n]$. Show that this construction defines a map of simplicial sets $g_X : \mathrm{sd}\, X \to X$ for all $X \in \mathbf{Set}_\Delta$. Show that pulling back along the g_{Δ^n} defines a natural transformation $\mathrm{id}_{\mathbf{Set}_\Delta} \Rightarrow \mathrm{Ex}$.

Problem 4. Let **2Cat** denote the category of 2-categories and strict 2-functors between them. Define a 2-category $\mathbb{\Delta}^n$ as follows:

- $\mathbb{\Delta}^n$ has one object i for each $i \in \{0, 1, \ldots, n\}$
- if $i \leq j$, there is a morphism $\phi_{i,j} : i \to j$, with $\phi_{i,i} = \mathrm{id}_i$
- For every sequence $i < i_1 < i_2 < \cdots < i_k < j$, there is a unique 2-morphism

$$\phi_{i_k,j} \circ \phi_{i_{k-1},i_k} \circ \cdots \circ \phi_{i_1,i_2} \circ \phi_{i,i_1} \Rightarrow \phi_{i,j}.$$

1. Show that the assignment

$$[n] \mapsto \mathbb{\Delta}^n$$

   defines a functor $D : \Delta \to \mathbf{2Cat}$.
2. Passing through the Yoneda embedding, D defines a functor

$$N_2 : \mathbf{2Cat} \to \mathbf{Set}_\Delta; \quad \mathcal{C} \mapsto \mathbf{2Cat}(D(-), \mathcal{C}).$$

   Given a 2-category $\mathcal{C}$, what data in $\mathcal{C}$ are needed to specify a 1-simplex of $N_2(\mathcal{C})$? What about 2- and 3-simplices?