Exercise sheet 2

Problem 1. A 2-category $\mathbb{C}$ consists of

- a set $\mathrm{ob}(\mathbb{C})$ of objects,
- for every pair (x, y) of objects, a category $\mathbb{C}(x, y)$ of morphisms from x to y,
- for every object x an object $\mathrm{id}_x \in \mathbb{C}(x, x)$ called the identity morphism,
- for every triple x, y, z, a functor

  $$\mu : \mathbb{C}(x, y) \times \mathbb{C}(y, z) \to \mathbb{C}(x, z)$$

  called composition law,

subject to the conditions

(1) for every object x, the functors

$$\mu(-, \mathrm{id}_y) : \mathbb{C}(x, y) \to \mathbb{C}(x, y)$$

and

$$\mu(\mathrm{id}_x, -) : \mathbb{C}(x, y) \to \mathbb{C}(x, y)$$

are the identity functors on $\mathbb{C}(x, y)$,

(2) for every 4-tuple (x, y, z, w), the diagram

$$\begin{tikzcd} \mathbb{C}(x,y) \times \mathbb{C}(y,z) \times \mathbb{C}(z,w) \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"] & \mathbb{C}(x,z) \times \mathbb{C}(z,w) \arrow[d, "\mu"] \\ \mathbb{C}(x,y) \times \mathbb{C}(y,w) \arrow[r, "\mu"] & \mathbb{C}(x,w) \end{tikzcd}$$

commutes.

Show that the set of small categories forms the objects of a 2-category $\mathbb{C}\mathrm{at}$ where, for small categories $\mathcal{C}$,$\mathcal{D}$, we define $\mathbb{C}\mathrm{at}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Problem 2. (1) Define the notion of an adjunction between two objects of a 2-category so that when applying your definition to the 2-category $\mathbb{C}\mathrm{at}$, you recover the notion of an adjunction between small categories as defined in class.

(2) Let $f : X \to Y$ be a map of sets. We obtain maps of posets

$$F : \mathcal{P}(X) \to \mathcal{P}(Y), U \mapsto f(U)$$

and

$$G : \mathcal{P}(Y) \to \mathcal{P}(X), V \mapsto f^{-1}(V).$$

Show that, interpreting these posets as categories, F and G define adjoint functors.

(3) For $n \geq 0$, we interpret the linearly ordered set $[n] = \{0, 1, \ldots, n\}$ as a category and hence as an object of the 2-category $\mathbb{C}\mathrm{at}$. For $0 \leq i \leq n$, we define the morphism

$$d_i : [n-1] \to [n], j \mapsto \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases}$$

and, for every $0 \leq i \leq n-1$, we define

$$s_i : [n] \to [n-1], j \mapsto \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

Show that, for every $0 \leq i \leq n-1$, there are adjunctions

$$d_{i+1} : [n-1] \longleftrightarrow [n] : s_i$$

and

$$s_i : [n] \longleftrightarrow [n-1] : d_i.$$

Problem 3. Let $(F, G, \eta, \varepsilon)$ be an adjunction between categories $\mathcal{C}$ and $\mathcal{D}$.

(1) Show that F is fully faithful if and only if the unit

$$\varepsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$$

is a natural isomorphism.

(2) Show that G is fully faithful if and only if the counit

$$\eta : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$$

is a natural isomorphism.

Problem 4. Let $\mathcal{C}$ be a locally small category, i.e., for every pair (x, y) of objects in $\mathcal{C}$, the set $\mathcal{C}(x, y)$ is small. We introduce the notation

$$\mathbf{Set}_{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}).$$

(1) Show that the formula on objects

$$H : \mathcal{C} \longrightarrow \mathbf{Set}_{\mathcal{C}},\ x \mapsto \mathcal{C}(-, x)$$

extends to a functor.

(2) Show that the functor H is fully faithful.

(3) Is H essentially surjective?