Exercise Sheet 13

Problem 1. Let $p : X \to S$ be an inner fibration.

(1) Show that, for an edge $f : x \to y$ of X, the following conditions are equivalent

(a) The map

$$X_{/f} \longrightarrow X_{/y} \times_{S_{p(y)}} S_{/p(f)}$$

is a trivial Kan fibration (i.e., f is p-Cartesian),

(b) For every $n \geq 2$, every lifting problem

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{h} & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & S \end{array}$$

such that h maps the edge $\Delta^{\{n-1,n\}}$ to the edge f, has a solution.

(2) Show that for $S = \Delta^0$, an edge in X is p-Cartesian if and only if it is an equivalence.

Problem 2. (1) Suppose $F : \mathcal{C} \to \mathcal{D}$ is a Cartesian fibration of categories. Show that $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ is a Cartesian fibration of ∞-categories.

(2) Let $\chi : \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$ be a functor, and $\pi : \mathcal{C}_\chi \to \mathcal{D}$ its contravariant Grothendieck construction. Show that the Cartesian morphisms in $\mathcal{C}_\chi$ are precisely those morphisms $(f, \phi) : (c, x) \to (d, y)$ such that $\phi : x \to \chi(f)(y)$ is an isomorphism.

Problem 3. Let k be a field and let **Vect** be the category with

- objects are given by pairs (X, E) where X is a topological space and $E \to X$ is a k-vector bundle on X,
- morphisms from (E, X) to (E', X') are given by commutative squares

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

where the restriction of φ to every fiber is k-linear.

(1) Show that the forgetful functor $\pi : \mathbf{Vect} \to \mathbf{Top}$ given by $(X, E) \mapsto X$ is a Cartesian fibration.

(2) Let X be a topological space, and let $\mathcal{U}$ be a collection of open subsets of X which is closed under intersection. We interpret $\mathcal{U}$ as poset with respect to inclusions of opens and consider the functor $f : \mathcal{U} \to \mathbf{Top}$, $U \mapsto U$. Let $\pi_{\mathcal{U}} : \mathbf{Vect}_{\mathcal{U}} \to \mathcal{U}$ denote the pullback of π along f.

(a) Explicitly describe the category $\mathrm{Fun}^{\#}_{\mathcal{U}}(\mathcal{U}, \mathbf{Vect}_{\mathcal{U}})$ of Cartesian sections of $\pi_{\mathcal{U}}$.

(b) Suppose that X is the union of all opens contained in $\mathcal{U}$. Show that there is an equivalence between $\mathrm{Fun}^{\#}_{\mathcal{U}}(\mathcal{U}, \mathbf{Vect}_{\mathcal{U}})$ and the category $\mathbf{Vect}_X$ of vector bundles over X (i.e. the fiber of π over X).

Problem 4. Let $p : X \to \Delta^1$ be an inner fibration with fibers given by the diagram

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & X & \longleftarrow & \mathcal{D} \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ \{0\} & \longrightarrow & \Delta^1 & \longleftarrow & \{1\} \end{array}$$

where both squares are pullback.

(1) Supposing p to be coCartesian fibration, construct an induced functor $f : \mathcal{C} \to \mathcal{D}$ of ∞-categories.

(2) Supposing p to be a Cartesian fibration, construct an induced functor $g : \mathcal{D} \to \mathcal{C}$ of ∞-categories.

(3) Show that if p is both a Cartesian fibration and a coCartesian fibration, then f and g define adjoint functors

$$hf : h\mathcal{C} \leftrightarrow h\mathcal{D} : hg$$

on homotopy categories.

(4) Suppose that p is both a Cartesian and a coCartesian fibration. Suppose that, for every edge e in X, we have that e is Cartesian if and only if e is coCartesian. Show that the functors f and g defined above are inverse equivalences of ∞-categories.