Hints for Probem 5 on Sheet 2

Hints.

1. Introduce

$$v_{0,j} := \begin{cases} \sum_{k=0}^{j}(-1)^k\binom{j}{k}\partial^k(e_{j-k}) & \text{for } 0 \leq j \leq n-1 \\ 0 & \text{else.} \end{cases}$$

   Further define recursively

$$v_{i+1,j} := \partial(v_{i,j}) + v_{i,j+1}.$$

2. Prove the following formulas by induction:

$$\partial^i(v) = \sum_{j=0}^{n-1}\frac{(t-\lambda)^j}{j!}v_{i,j} \tag{1}$$

   and

$$v_{i,j} = \sum_{k=0}^{j}(-1)^k\binom{j}{k}\partial^k(e_{i+j-k}). \tag{2}$$

3. Set, for a formal variable T,

$$e_i(T) := \sum_{j=0}^{n-1}\frac{T^j}{j!}v_{i,j} \in F[T] \otimes_F M$$

   and use the formulas (1) and (2) to deduce $e_i(0) = e_i$ and $e_i(t-\lambda) = \partial^i(v)$.

4. Argue that there exists a polynomial $f(T) \in F[T]$ such that

$$e_0(T) \wedge \cdots \wedge e_{n-1}(T) = f(T)e_0 \wedge \cdots \wedge e_{n-1} \in \bigwedge_{F[T]}^{n} F[T] \otimes_F M$$

   such that $f(T) \neq 0$ and $\deg(f) \leq n(n-1)$. Deduce that $f(t-\lambda)$ can only be zero for finitely many $\lambda \in K_F$.

5. Conclude by showing that $e_0(t-\lambda), \ldots, e_{n-1}(t-\lambda)$ are F-linearly independent for those $\lambda \in K_F$ such that $f(t-\lambda) \neq 0$.

□