Exercise Sheet 7

Problem 1. Let K be a field and consider the K-vector space $K^{n\times n}$ of matrices. Note that $K^{n\times n}$ also admits a K-algebra structure given by matrix multiplication. Show as follows that a K-linear automorphism of $K^{n\times n}$ is an algebra automorphism if and only if it is given by conjugation with a matrix $A \in \mathrm{GL}(n, K)$, i.e.,

$$K^{n\times n} \longrightarrow K^{n\times n}, X \mapsto A^{-1}XA.$$

1. Let $\varphi : F^{n\times n} \to F^{n\times n}$ be a K-algebra automorphism. Choose $u, w \in K^n$ nonzero and use the injectivity of φ to find $z \in K^n$ such that $\varphi(uw^{\mathrm{tr}})z \neq 0$.
2. Define $A : K^n \to K^n, v \mapsto \varphi(vw^{\mathrm{tr}})z$ and show that A is K-linear.
3. Show that, for every $X \in K^{n\times n}$ and every $y \in K^n$, we have

   $$AXy = \varphi(X)Ay$$

   and deduce that $AX = \varphi(X)A$.
4. Let $x \in K^n$. Use that φ is surjective and $Au \neq 0$ to show that we may find $B \in K^{n\times n}$ such that $\varphi(B)Au = x$. Deduce from $ABu = \varphi(B)Au$ that A is surjective and hence invertible.

Problem 2. Let K be an algebraically closed field and let $Z \subset \mathrm{GL}(n, K)$ be the subgroup of scalar matrices, i.e., matrices of the form λI_n with $\lambda \in K^\times$. We denote by

$$\mathrm{PGL}(n, K) = \mathrm{GL}(n, K)/Z$$

the quotient group. Show that $\mathrm{PGL}(n, K)$ is a linear algebraic group. Hint: Consider the group homomorphism

$$\mathrm{GL}(n, K) \longrightarrow \mathrm{GL}(K^{n\times n}), A \mapsto (X \mapsto A^{-1}XA),$$

use Problem 1, and further show that the condition for $\varphi \in \mathrm{GL}(K^{n\times n})$ to be an algebra automorphism is given by polynomial equations.

Problem 3. Let R/F be a Picard-Vessiot ring and let $G = \mathrm{Gal}^\partial(R/F)$ be the ∂-Galois group, considered as a linear algebraic group, and let $\mathfrak{g}$ denote the Lie algebra of G. Show that there is a canonical isomorphism of Lie algebras

$$\mathfrak{g} \cong \mathrm{Der}^\partial(R/F)$$

where $\mathrm{Der}^\partial(R/F)$ denotes the Lie algebra of F-linear derivations of R which commute with the given derivation ∂.

Problem 4. Let R/F be a Picard-Vessiot ring for $A \in F^{n\times n}$. Let $H \subseteq \mathrm{GL}(n, K)$ be a Zariski-closed subgroup with Lie algebra $\mathfrak{h} \subseteq \mathfrak{gl}_n$. Suppose that $A \in \mathfrak{h} \otimes_K F$. Show that there exists a choice of fundamental solution matrix $Y \in \mathrm{GL}(n, R)$ such that, with respect to the corresponding embedding $\mathrm{Gal}^\partial(R/F) \subseteq \mathrm{GL}(n, K)$, we have $\mathrm{Gal}^\partial(R/F) \subseteq H$.