Exercise Sheet 4

Problem 1. Let K be an algebraically closed field of characteristic 0, let $K[[t]]$ be the ring of power series and $K((t))$ its field of fractions. Let $A \in K[t]^{n\times n}$.

1. Show that there exists a fundamental solution matrix $Y \in \mathrm{GL}(n, K[[t]])$ of the form
$$Y = I + \sum_{k>0} B_k t^k$$
with $B_k \in K^{n\times n}$.

2. Deduce that there exists a Picard-Vessiot extension $E/K(t)$ for A with $E \subset K((t))$.

Problem 2. Let F be a ∂-field of characteristic 0 with algebraically closed field of constants K and let $a \in F^*$. Consider the matrix
$$A = (a) \in F^{1\times 1}.$$
and let
$$R = F[T, T^{-1}]$$
with $\partial(T) = aT$. Suppose that there exists $n \in \mathbb{N} \setminus \{0\}$ and $0 \neq y \in F$ with $\partial(y) = nay$. We assume $n > 0$ to be minimal. Let
$$\overline{R} = R/(T^n - y)$$
be the Picard-Vessiot ring for A (cf. Sheet 3) and E its field of fractions. Determine the ∂-Galois group $\mathrm{Gal}^\partial(E/F)$ and verify, in particular, that it is the solution set of polynomial equations on $\mathrm{GL}(1, K)$.

Problem 3. Let F be a ∂-field of characteristic 0 with algebraically closed field of constants K, and let $A \in F^{n\times n}$. Let G be the ∂-Galois group of A considered as a subgroup $G \leq \mathrm{GL}(n, K)$ via the choice of a generating fundamental solution matrix $Y \in \mathrm{GL}(n, E)$ for a Picard-Vessiot extension E/F. Show that $G \leq \mathrm{SL}(n, K)$ if and only if the ∂-equation $\partial(u) = \mathrm{tr}(A)u$ has a nonzero solution in F.

Problem 4. Let M be a ∂-field of characteristic 0 with algebraically closed field of constants K. Let $n \geq 1$, and consider the polynomial ring (in infinitely many variables)
$$R = M[y_1^{(0)}, y_1^{(1)}, y_1^{(2)}, \ldots, y_2^{(0)}, y_2^{(1)}, y_2^{(2)}, \ldots, y_n^{(0)}, y_n^{(1)}, y_n^{(2)}, \ldots]$$
equipped with the derivation ∂ given by extending the formula $\partial(y_j^{(i)}) = y_j^{(i+1)}$ via additivity and Leibniz rule to R. We set $y_j := y_j^{(0)}$. Let E be the field of fractions of R with derivation extended from R via the quotient rule.

1. Show that $K_E = K$.

2. Let $l \in E[\partial]$ be the differential operator obtained by formally evaluating the expression
$$l(y) = \frac{\mathrm{wr}(y, y_1, \ldots, y_n)}{\mathrm{wr}(y_1, \ldots, y_n)} = \partial^n(y) + a_{n-1}\partial^{n-1}(y) + \cdots + a_0 y.$$
Show that
$$a_{n-1} = \frac{\partial(\mathrm{wr}(y_1, \ldots, y_n))}{\mathrm{wr}(y_1, \ldots, y_n)}.$$

3. Let $F \subset E$ be the smallest subfield of E containing M and the coefficients a_k, $0 \leq k \leq n-1$. Show that, for every $C \in \mathrm{GL}(n, K)$, the formula

$$(\sigma_C(y_1), \dots, \sigma_C(y_n)) := (y_1, \dots, y_n)C$$

extends to define an automorphism σ_C of the ∂-field E fixing F.

4. Show that E/F is a Picard-Vessiot extension with ∂-Galois group $\mathrm{GL}(n, K)$.