Homological Algebra - Problem Set 7

Problem 1. Let K be the smallest simplicial complex on $\{0, 1, \ldots, N\}$ which contains the collection of subsets

$$\{\{0,1\}, \{1,2\}, \{2,3\}, \ldots, \{N-1, N\}, \{N, 0\}\}.$$

Let $G = \mathbb{Z}/(N+1)\mathbb{Z}$ be the cyclic group of order $N+1$. Show that the natural action of G on $\{0, 1, \ldots, N\}$ induces a free G-action on K. Show that K is a simplicial 1-dimensional homology sphere and compute the corresponding 2-periodic $\mathbb{Z}G$-free resolution of $\mathbb{Z}$. Compare this to results obtained in class.

Problem 2. Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H} \cong \mathbb{R}^4$.

(1) The elements of Q_8, interpreted as vectors in $\mathbb{R}^4$, form the vertices of a 4-dimensional simplicial polytope called the *hexadecachoron* (or 16-*cell*). The hexadecachoron is defined to be the convex hull of Q_8 in $\mathbb{R}^4$. Show that its boundary gives rise to a simplicial complex K on the set underlying the group Q_8 which has 16 tetrahedra, 32 triangles, 24 edges, and 8 vertices. Show that K is a simplicial 3-dimensional homology sphere (For this you may cite a result from algebraic topology or use a computer). Show that Q_8 acts freely on K. Deduce that Q_8 has 4-periodic group homology.

(2) Use the simplicial complex K to explicitly construct a 4-periodic free resolution of the trivial $\mathbb{Z}Q_8$-module $\mathbb{Z}$ and show that

$$H_i(Q_8, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{for } i \equiv 1 (\operatorname{mod} 4), \\ 0 & \text{for } i \text{ even and nonzero}, \\ \mathbb{Z}/8\mathbb{Z} & \text{for } i \equiv 3 (\operatorname{mod} 4). \end{cases}$$

Problem 3. Let G be a group.

(1) Assume that G is a nontrivial finite cyclic group. Show that the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ does not admit a projective resolution of finite length.

(2) Assume that G has torsion. Show that the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ does not admit a projective resolution of finite length.

(3) Deduce that if G has torsion, then the space $K(G,1)$ cannot be realized as a CW-complex with finitely many cells.

Problem 4. Let G be an abelian group and let $B_\bullet$ denote its bar construction. Given integers $p \geq 0$ and $q \geq 0$, a (p,q)-*shuffle* is a permutation σ of the set $\{1, 2, \ldots, p+q\}$ satisfying the conditions $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$. We define the *shuffle product* on the abelian group $B_* = \oplus_n B_n$ by bilinearly extending the formula

$$a[g_1|\ldots|g_p] * b[g_{p+1}|\ldots|g_{p+q}] = \sum_\sigma (-1)^{\mathrm{sign}(\sigma)} ab[g_{\sigma^{-1}(1)}|g_{\sigma^{-1}(2)}|\ldots|g_{\sigma^{-1}(p+q)}]$$

where σ runs over all (p,q)-shuffles.

(1) Show that the shuffle product is unital and associative so that B_* becomes a graded ring.

(2) Show that, for $x \in B_p$ and $y \in B_q$, we have

$$x * y = (-1)^{pq} y * x.$$

A graded ring with this property is called *graded-commutative.*

(3) Show that, for $x \in B_p$ and $y \in B_q$, we have the Leibniz rule

$$d(x * y) = d(x) * y + (-1)^p x * d(y).$$

(4) Let R be a commutative $\mathbb{Z}G$-algebra. Show that $H_*(G,R)$ is a graded-commutative ring.