Homological Algebra - Problem Set 6

Problem 1. Let k be a field, Q a quiver, and A the path algebra of Q. For $i \geq 0$, determine the dimension of the k-vector space

$$\mathrm{Tor}_i^{A\otimes_k A^{\mathrm{op}}}(A, A).$$

Problem 2. Let G be a group and let $\mathbb{Z}G$ the corresponding group ring. Define the augmentation map

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z}, \sum \lambda_g g \mapsto \sum \lambda_g$$

and the augmentation ideal $\mathcal{I} \subset \mathbb{Z}G$ to be the kernel of ε.

(1) Show that, if G is infinite, then $(\mathbb{Z}G)^G = 0$.

(2) Show that, if G is finite, then the augmentation ideal is the kernel of the map $\mathbb{Z}G \to \mathbb{Z}G$ given by multiplication with the norm $N = \sum_{g\in G} g$.

(3) Show that the map $\theta : G \to \mathcal{I}/\mathcal{I}^2$ defined by $\theta(g) = g - 1$ is a group homomorphism which determines an isomorphism $G/[G, G] \cong \mathcal{I}/\mathcal{I}^2$.

Problem 3. (1) Let G be a group, and let A be a trivial G-module. Show that $H^1(G, A) \cong \mathrm{Hom}_{\mathbb{Z}}(G/[G, G], A)$.

(2) Let G be a finite group. Consider the abelian groups $\mathbb{Z}, \mathbb{C}$ and $\mathbb{C}^*$ as trivial G-modules. Show that $H^1(G, \mathbb{Z}) = H^1(G, \mathbb{C}) = 0$. Show that $H^2(G, \mathbb{Z})$ is isomorphic to $H^1(G, \mathbb{C}^*) \cong \mathrm{Hom}_{\mathrm{Groups}}(G, \mathbb{C}^*)$, the group of 1-dimensional complex representations of G.

Problem 4. (1) Let $X_\bullet$ be a complex of free abelian groups, and let A be an abelian group. Show that there is an exact sequence

$$0 \longrightarrow H_n(X) \otimes_{\mathbb{Z}} A \longrightarrow H_n(X \otimes_{\mathbb{Z}} A) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), A) \longrightarrow 0.$$

Hint. Consider the exact sequences $0 \longrightarrow Z_n \longrightarrow X_n \longrightarrow d(X_n) \longrightarrow 0$ of free (!) abelian groups.

(2) Show that the exact sequence from (1) splits (noncanonically).

(3) Let G be a group and A a trivial G-module. Show that, for every $n \geq 1$, there is a noncanonical isomorphism

$$H_n(G, A) \cong H_n(G, \mathbb{Z}) \otimes_{\mathbb{Z}} A \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(G, \mathbb{Z}), A).$$

(4) Let G be a group and A a trivial G-module. Show that, for every $n \geq 1$, there is a noncanonical isomorphism

$$H^n(G, A) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(G, \mathbb{Z}), A) \oplus \mathrm{Ext}_{\mathbb{Z}}^1(H_{n-1}(G, \mathbb{Z}), A).$$