Homological Algebra - Problem Set 5

Problem 1. An abelian group A is called divisible, if, for every $a \in A$ and every nonzero natural number n, there exists $b \in A$ such that $nb = a$.

(1) Show that the abelian groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible.

(2) Show that an abelian group I is an injective object of **Ab** if and only if it is divisible. (Hint: Given a divisible group I and an inclusion $A \subset B$ and a map $g : A \to I$, to show that g extends along $A \subset B$, consider the partially ordered set of intermediate extensions of g to A' with $A \subset A' \subset B$ and apply Zorn's lemma).

(3) Show that the category **Ab** of abelian groups has enough injectives. (Hint: For a given abelian group A, consider the morphism

$$i : A \to \prod_{\mathrm{Hom}_{\mathbf{Ab}}(A,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$$

(4) Let R be a ring. Show that the category **mod**$-\,R$ of right R-modules has enough injectives. (Hint: The abelian group $\mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ has a natural right R-module structure.)

Problem 2. Assume $\mathcal{A}$ has enough projectives and enough injectives. Show that Ext is balanced: for objects A, B, there exists, for every n, a canonical isomorphism of abelian groups

$$R^n \mathrm{Hom}_{\mathcal{A}}(A, -)(B) \cong R^n \mathrm{Hom}_{\mathcal{A}}(-, B)(A).$$

Problem 3. Let k be a field.

(1) Let Q be the quiver with one vertex 0 and one arrow ρ starting and ending at 0. Given $\lambda \in k$, let $V(\lambda)$ denote the k-linear representation of Q with $V_0 = k$ and the endomorphism of k corresponding to ρ given by multiplication by λ. For $\lambda, \lambda' \in k$, determine all extension classes of $V(\lambda)$ by $V(\lambda')$.

(2) Let Q be any finite quiver. For a vertex i of Q, we denote by $S(i)$ the k-linear representation of Q given by

$$S(i)_j = \begin{cases} k & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

For vertices i,j of Q, determine $\mathrm{Ext}^1(S(i), S(j))$.