Homological Algebra - Problem Set 0 (Warmup)

Problem 1. Let $C_n = \mathbb{Z}/8\mathbb{Z}$, $n \in \mathbb{Z}$. Let $d : C_n \to C_{n-1}$ be the map induced by multiplication by 4 considered modulo 8. Show that $d^2 = 0$ and compute the homology groups of the resulting complex of abelian groups.

Problem 2. Find a free resolution of the $\mathbb{C}[x, y]$-module $\mathbb{C}[x, y]/(x, y)$.

Problem 3. For each of the following simplicial complexes determine explicity the simplicial chain complex with coefficients in $\mathbb{R}$ and compute its homology.

(1) The simplicial complex on $\{0, 1, 2, 3\}$ given by all subsets.

(2) The simplicial complex on $\{0, 1, 2, 3\}$ given by all subsets of cardinality ≤ 2.

Problem 4. Let k be a field and let A be an associative k-algebra. We define, for every $n \geq 0$, the k-vector space

$$C_n(A) = \underbrace{A \otimes_k A \otimes_k \cdots \otimes_k A}_{n+1 \text{ copies}}.$$

Further, we define maps $d : C_n(A) \to C_{n-1}(A)$ by k-linearly extending the formula

$$\begin{aligned} d(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = {} & (a_0 a_1) \otimes a_2 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n \\ & + (-1)^n (a_n a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Show that $d^2 = 0$ so that $C_\bullet(A)$ forms a chain complex of vector spaces. Explicitly determine the complex $C_\bullet(k)$ and compute its homology.

Problem 5. Let R be a ring and let $f : C \to D$ be a morphism of chain complexes of R-modules.

(1) Show that f preserves cycles and boundaries and hence induces, for every n, a map $H_n(f) : H_n(C) \to H_n(D)$ of homology modules. Show that this defines, for every n, a functor from the category of chain complexes of R-modules to the category of R-modules.

(2) Recall that f is called a quasi-isomorphism if, for every n, the map $H_n(f)$ is an isomorphism. A morphism $g : D \to C$ of chain complexes is called a *quasi-inverse* of f if, for every n, the map $H_n(g)$ is the inverse of $H_n(f)$. Give an explicit example of a quasi-isomorphism of chain complexes of abelian groups which does not admit a quasi-inverse (provide a proof).

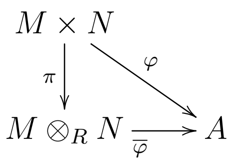
Problem 6. Let R be a ring, M a right R-module, N a left R-module, and A an abelian group. A map of sets $\varphi : M \times N \to A$ is called *bilinear* if the following hold

1. for $m_1, m_2 \in M$, $n \in N$, we have $\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$,

2. for $m \in M$, $n_1, n_2 \in N$, we have $\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$,

3. for $r \in R$, $m \in M$, $n \in N$, we have $\varphi(mr, n) = \varphi(m, rn)$.

(1) Explicitly construct an abelian group $M \otimes_R N$ equipped with a bilinear map $\pi : M \times N \to M \otimes_R N$ which is *universal* in the following sense: For every bilinear map $\varphi : M \times N \to A$ there exists a unique homomorphism $\overline{\varphi} : M \otimes_R N \to A$ of abelian groups such that the diagram



commutes.

(2) Show that the universal property in (1) uniquely determines the abelian group $M \otimes_R N$ up to isomorphism. We call $M \otimes_R N$ the *tensor product of M and N over R.*

(3) Let m, n be integers. Determine the tensor product $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$.