On large deviation for extremes

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Summary. Recently a weighted approximation for the tail empirical distribution function has been developed (Drees et al. (2002)). We show that the same result can also be used to improve a known uniform approximation of the distribution of the maximum of a random sample. From this a general result about large deviations of this maximum is derived. In addition, the relationship between two second order conditions used in extreme value theory is clarified.

1 Introduction

Let $\{X_n,\, n \geq 1\}$ be independent identically distributed random variables with common distribution function $F(x)$. Suppose F is in the domain of attraction of the extreme value distribution with index $\gamma \in \mathbb{R}$

$$G_\gamma(x) := \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \quad 1+\gamma x \geq 0,$$

that is, there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$P(M_n \leq a_n x + b_n) \to G_\gamma(x), \qquad x \in \mathbb{R}, \tag{1.1}$$

as $n \to \infty$, where $M_n := \max(X_1, X_2, ..., X_n)$.

Since the limit function G_γ is continuous, we have

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F^n(a_n x + b_n) - G_\gamma(x)| = 0.$$

Cheng and Jiang (2001) proved that under the second order strengthening (1.4) of condition (1.1) one can find a sequence $A(n)$ satisfying $A(n) \to 0$, as $n \to \infty$ and A is regularly varying with index $\rho \leq 0$ and normalizing constants $\tilde{a}_n > 0$ and $\tilde{b}_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \sup_{1+\gamma x>0} \left| \frac{F^n(\tilde{a}_n x + \tilde{b}_n) - G_\gamma(x)}{A(n)} + (1+\gamma x)^{-1/\gamma-1} G_\gamma(x) \bar{H}_{\gamma,\rho}((1+\gamma x)^{1/\gamma}) \right| = 0, \tag{1.2}$$

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where the function $\bar{H}_{\gamma,\rho}$ is defined in (3.1) below. De Haan and Resnick (1996) established a similar approximation under a somewhat stronger second order condition. We will show that, under the same condition but with slightly different normalizing constants, a weighted version of this result holds, that is more accurate for values of x close to the right endpoint $1/(-\gamma) \vee 0)$ of G_γ, that is, ∞ if $\gamma \geq 0$ and $1/(-\gamma)$ if $\gamma < 0$.

From this result it is easily deduced that

$$\lim_{n\to\infty} \frac{1 - F^n(a_n x_n + b_n)}{1 - G_\gamma(x_n)} = 1 \tag{1.3}$$

for all sequences $x_n \uparrow 1/((-\gamma) \vee 0)$. Convergence (1.3), which can be considered a result about large deviations of the maximum M_n from its 'typical' behavior, was also studied in Section 2.3 of the monograph by Resnick (1987). There, for different normalizing constants a_n and b_n, quite complicated necessary and sufficient conditions on the maximal rate at which x_n may tend to $1/((-\gamma) \vee 0)$ were given such that (1.3) holds. In contrast, for our choice of the normalizing constants, the large deviations result (1.3) holds for *all* sequences $x_n \uparrow 1/((-\gamma) \vee 0)$, provided the second order condition (1.4) is met with $\rho < 0$.

Condition (1.1) is equivalent to the existence of a positive function a^* such that

$$\lim_{t\to\infty} \frac{V(tx) - V(t)}{a^*(t)} = \frac{x^\gamma - 1}{\gamma}$$

for all $x > 0$, where the function V is defined as a generalized inverse:

$$V(t) := \left(\frac{1}{-\log F}\right)^{\leftarrow}(t) = F^{\leftarrow}(\mathrm{e}^{-1/t}).$$

Cheng and Jiang (2001) proved that the following second order condition is necessary for a uniform approximation of type (1.2): there exists a (positive or negative) function A^* and a parameter $\rho \leq 0$ such that for all $x > 0$

$$\lim_{t\to\infty} \frac{\frac{V(tx)-V(t)}{a^*(t)} - \frac{x^\gamma - 1}{\gamma}}{A^*(t)} = H_{\gamma,\rho}(x) := \begin{cases} \frac{x^{\gamma+\rho}-1}{\gamma+\rho}, & \rho < 0,\ \gamma+\rho \neq 0, \\ \log x, & \rho < 0,\ \gamma+\rho = 0, \\ \frac{1}{\gamma} x^\gamma \log x, & \rho = 0 \neq \gamma, \\ \frac{1}{2}\log^2 x, & \rho = 0 = \gamma. \end{cases} \tag{1.4}$$

Then there exist functions a and A satisfying $a(t) \sim a^*(t)$ and $A(t) \sim A^*(t)$ as $t \to \infty$, such for all $\epsilon > 0$ there exists a constant $t_\epsilon > 0$ such that for all t, $tx \geq t_\epsilon$

$$x^{-(\gamma+\rho)} e^{-\epsilon|\log x|} \left| \frac{\frac{V(tx)-V(t)}{a(t)} - \frac{x^\gamma-1}{\gamma}}{A(t)} - H_{\gamma,\rho}(x) \right| < \epsilon \tag{1.5}$$

(see Drees (1998)). Cheng and Jiang (2001) gave explicit representations of the functions a and A in terms of F. Under this second order condition, following the lines of Drees et al. (2002), one may prove a weighted approximation to the tail of the empirical distribution function which will be central for the proof of our main result.

While here we work with a second order condition for the function V, usually the analogous condition for $U := (1/1-F)^{\leftarrow}$ is considered. The relationship between these two conditions is clarified in the Appendix.

2 Main results

Our main result is a weighted approximation to the normalized distribution function $F^n(a_n x + b_n)$ of the maximum M_n where the additive constant b_n is chosen equal to $V(n)$.

Theorem 2.1. *Assume that V satisfies (1.4) with $\gamma \neq 0$ or $\rho < 0$. Define for $n \in \mathbb{N}$*

$$\begin{aligned} a_n &:= \begin{cases} a(n)\Big(1 + \frac{\gamma}{\gamma+\rho} A(n)\Big) & \text{if } \rho < 0, \gamma + \rho \neq 0, \\ a(n) & \text{otherwise,} \end{cases} \\ b_n &:= V(n) \end{aligned}$$

and

$$\tilde{H}_{\gamma,\rho}(x) := \begin{cases} \frac{x^{\gamma+\rho} - x^\gamma}{\gamma+\rho} & \text{if } \rho < 0, \gamma+\rho \neq 0, \\ H_{\gamma,\rho}(x) & \text{otherwise.} \end{cases}$$

Then for all x_0 greater than the left endpoint $-\frac{1}{\gamma \vee 0}$ of G_γ and all $\epsilon > 0$

$$\begin{aligned} &\sup_{x_0 \le x < \frac{1}{(-\gamma)\vee 0}} \Big((1+\gamma x)^{1/\gamma}\Big)^{1-\epsilon I_{\{\rho=0\}}} \\ &\times \left| \frac{F^n(a_n x + b_n) - G_\gamma(x)}{A(n)} + G_\gamma(x)(1+\gamma x)^{-1/\gamma - 1} \tilde{H}_{\gamma,\rho}((1+\gamma x)^{1/\gamma}) \right| = o(1) \end{aligned}$$

as $n \to \infty$.

The proof of Theorem 2.1 is based on a similar approximation where a less natural additive constant $\bar{b}_n$ and the scaling constant $\bar{a}_n = a(n)$ are used (see Proposition 3.1). At first glance, seemingly one has to pay for the natural choice $b_n = V(n)$ by a more complicated scaling constant a_n. However, (1.5) also holds when $a(t)$ is replaced with $a(t)(1 + \gamma/(\gamma+\rho)A(t))$ in the case $\rho < 0, \gamma + \rho \neq 0$ and $H_{\gamma,\rho}$ is replaced with $\tilde{H}_{\gamma,\rho}$. Hence there is nothing special about the normalizing function a (and hence also about $\bar{a}_n = a(n)$), but its particular form is only due to the quite arbitrary choice of the limiting function $H_{\gamma,\rho}$ often considered in the literature.

From the weighted approximation established in Theorem 2.1, results on the relative error of the extreme value approximation of F^n and on large deviations follow readily:

Corollary 2.1. *Under the conditions of Theorem 2.1 with $\rho < 0$ one has*

$$\sup_{x_0 \le x < \frac{1}{(-\gamma)\vee 0}} \left| \frac{\frac{1-F^n(a_n x + b_n)}{1 - G_\gamma(x)} - 1}{A(n)} - \frac{G_\gamma(x)}{1 - G_\gamma(x)} (1+\gamma x)^{-1/\gamma - 1} \tilde{H}_{\gamma,\rho}((1+\gamma x)^{1/\gamma}) \right| = o(1)$$

as $n \to \infty$. In particular,

$$\lim_{n\to\infty} \frac{1 - F^n(a_n x_n + b_n)}{1 - G_\gamma(x_n)} = 1$$

for any sequence $x_n \uparrow 1/((-\gamma) \vee 0)$.

3 Proofs

The main ingredient of the proof of Theorem 2.1 is an approximation similar to the one asserted in Theorem 2.1 but using different normalizing constants and, as a consequence, the following modification of the limiting function:

$$\bar{H}_{\gamma,\rho}(x) := \begin{cases} \frac{x^{\gamma+\rho}}{\gamma+\rho} & \text{if } \rho < 0, \gamma+\rho \neq 0, \\ H_{\gamma,\rho}(x) & \text{otherwise.} \end{cases} \tag{3.1}$$

Proposition 3.1. *Let*

$$\begin{aligned} \bar{a}_n &:= a(n), \\ \bar{b}_n &:= \begin{cases} V(n) - \frac{1}{\gamma+\rho} a(n) A(n) & \textit{if } \rho < 0, \gamma+\rho \neq 0, \\ V(n) & \textit{otherwise.} \end{cases} \end{aligned}$$

Then, under the conditions of Theorem 2.1,

$$\sup_{x_0 \le x < \frac{1}{(-\gamma)\vee 0}} \left((1+\gamma x)^{1/\gamma} \right)^{1-\rho-\epsilon} \\ \times \left| \frac{F^n(\bar{a}_n x + \bar{b}_n) - G_\gamma(x)}{A(n)} + G_\gamma(x)(1+\gamma x)^{-1/\gamma-1} \bar{H}_{\gamma,\rho}((1+\gamma x)^{1/\gamma}) \right| = o(1)$$

as $n \to \infty$.

Proof:
By the very same arguments as used in the proof of the Propositions 3.1 and 3.2 by Drees et al. (2002), one obtains

$$\sup_{x_0 \le x < \frac{1}{0 \vee (-\gamma)}} w(x) \Big| \frac{n(-\log F(\bar{a}_n x + \bar{b}_n)) - (1+\gamma x)^{-1/\gamma}}{A(n)} \\ - (1+\gamma x)^{-1/\gamma-1} \bar{H}_{\gamma,\rho}((1+\gamma x)^{1/\gamma}) \Big| = o(1)$$

with $w(x) := \left((1+\gamma x)^{1/\gamma} \right)^{1-\rho-\epsilon}$. This implies

$$\begin{aligned} F^n(\bar{a}_n x + \bar{b}_n) &= \exp\left(n \log F(\bar{a}_n x + \bar{b}_n) \right) \\ &= \exp\left(-(1+\gamma x)^{-1/\gamma} - A(n)\Phi(x) - o(1)\frac{A(n)}{w(x)} \right) \\ &= G_\gamma(x) \exp\left(-A(n)\Phi(x) - o(1)\frac{A(n)}{w(x)} \right) \end{aligned} \tag{3.2}$$

as $n \to \infty$, where the $o(1)$-term is uniform for $x_0 \le x < 1/((-\gamma) \vee 0)$ and

$$\Phi(x) := (1+\gamma x)^{-1/\gamma - 1} \bar{H}_{\gamma,\rho}((1+\gamma x)^{1/\gamma}).$$

Note that as $x \uparrow 1/((-\gamma) \vee 0)$

$$\Phi(x) = \begin{cases} \frac{(1+\gamma x)^{(\rho-1)/\gamma}}{\gamma+\rho}, & \rho < 0, \gamma+\rho \neq 0 \\ (1+\gamma x)^{-1/\gamma-1} \log((1+\gamma x)^{1/\gamma}), & \rho < 0, \gamma+\rho = 0 \\ \frac{1}{\gamma}(1+\gamma x)^{-1/\gamma} \log((1+\gamma x)^{1/\gamma}), & \rho = 0 \neq \gamma \end{cases} = o\left(\frac{1}{w(x)}\right). \tag{3.3}$$

Thus

$$A(n)\Phi(x) + o(1)\frac{A(n)}{w(x)} = O(1)\frac{A(n)}{w(x)} \longrightarrow 0 \tag{3.4}$$

as $n \to \infty$ uniformly in x, provided we choose $\epsilon < 1 - \rho$. Because of

$$1 - x \le e^{-x} \le 1 - x + x^2$$

for $-1 \le x \le 1$, we have eventually

$$\begin{aligned} &1 - A(n)\Phi(x) - o(1)\frac{A(n)}{w(x)} \\ &\le \exp\left(-A(n)\Phi(n) - o(1)\frac{A(n)}{w(x)}\right) \\ &\le 1 - A(n)\Phi(x) - o(1)\frac{A(n)}{w(x)} + \left(A(n)\Phi(x) + o(1)\frac{A(n)}{w(x)}\right)^2. \end{aligned}$$

By (3.4) the squared term is of smaller order than $A(n)/w(x)$ uniformly in x. Hence, in view of (3.2),

$$F^n(\bar{a}_n x + \bar{b}_n) = G_\gamma(x)\left(1 - A(n)\Phi(x) - o(1)\frac{A(n)}{w(x)}\right)$$

as $n \to \infty$, uniformly for $x_0 \le x < 1/((-\gamma) \vee 0)$. □

For the proof of Theorem 2.1 we need two additional lemmas. Define

$$\tilde{x} := x + \Delta_x A(n) \quad \text{with} \quad \Delta_x := \frac{1+\gamma x}{\gamma+\rho} I_{\{\rho<0,\gamma+\rho\neq 0\}}. \tag{3.5}$$

Lemma 3.1.

$$\sup_{x_0 \le x < \frac{1}{(-\gamma)\vee 0}} \left| \frac{\frac{(1+\gamma\tilde{x})^{-1/\gamma}}{(1+\gamma x)^{-1/\gamma}} - 1}{A(n)} + \frac{1}{\gamma+\rho} I_{\{\rho<0,\gamma+\rho\neq 0\}} \right| = o(1).$$

Proof: Suppose $\gamma \neq 0$. Since $\Delta_x/(1+\gamma x)$ is constant, a Taylor expansion yields

$$\begin{aligned}
(1+\gamma\tilde{x})^{-1/\gamma} &= (1+\gamma x+\gamma\Delta_x A(n))^{-1/\gamma} \\
&= (1+\gamma x)^{-1/\gamma}\left(1+\gamma\frac{\Delta_x}{1+\gamma x}A(n)\right)^{-1/\gamma} \\
&= (1+\gamma x)^{-1/\gamma}\left(1-\frac{\Delta_x A(n)}{1+\gamma x}+o(A(n))\right)
\end{aligned}$$

and hence the assertion. The proof is similar in the case $\gamma = 0$. □

Next define for the ease of writing

$$y := (1+\gamma x)^{1/\gamma} \quad \text{and} \quad \tilde{y} := (1+\gamma\tilde{x})^{1/\gamma}.$$

Then Lemma 3.1 can be reformulated as follows: for all $y_0 > 0$

$$\lim_{n\to\infty}\sup_{y\geq y_0}\left|\frac{y/\tilde{y}-1}{A(n)}+\frac{1}{\gamma+\rho}I_{\{\rho<0,\gamma+\rho\neq 0\}}\right| = 0. \tag{3.6}$$

Lemma 3.2.

$$\lim_{n\to\infty}\sup_{y\geq y_0} y\left|\frac{\mathrm{e}^{-1/\tilde{y}}-\mathrm{e}^{-1/y}}{A(n)}+\frac{\mathrm{e}^{-1/y}}{y}\cdot\frac{1}{\gamma+\rho}I_{\{\rho<0,\gamma+\rho\neq 0\}}\right| = 0.$$

Proof: By (3.6) one obtains

$$\begin{aligned}
\mathrm{e}^{-1/\tilde{y}}-\mathrm{e}^{-1/y} &= \mathrm{e}^{-1/y}\left(\exp\left(-\frac{y/\tilde{y}-1}{y}\right)-1\right) \\
&= \mathrm{e}^{-1/y}\left(-\frac{A(n)}{y(\gamma+\rho)}I_{\{\rho<0,\gamma+\rho\neq 0\}}(1+o(1))\right),
\end{aligned}$$

where the $o(1)$-term is uniform in x, and thus the assertion. □

Proof of Theorem 2.1: In view of Lemma 3.1, Proposition 3.1 implies that for all $x_0 > -1/(0 \vee \gamma)$

$$\begin{aligned}
&\sup_{x_0\leq\tilde{x}<\frac{1}{(-\gamma)\vee 0}}\left((1+\gamma\tilde{x})^{1/\gamma}\right)^{1-\rho-\epsilon} \\
&\times\left|\frac{F^n(\bar{a}_n\tilde{x}+\bar{b}_n)-G_\gamma(\tilde{x})}{A(n)}+G_\gamma(\tilde{x})(1+\gamma\tilde{x})^{-1/\gamma-1}\bar{H}_{\gamma,\rho}((1+\gamma\tilde{x})^{1/\gamma})\right| = o(1).
\end{aligned}$$

Next note that $\bar{a}_n\tilde{x}+\bar{b}_n = a_n x+b_n$, so that the last approximation can be rewritten as

$$\sup_{\tilde{y}\geq y_0}\tilde{y}^{1-\rho-\epsilon}\left|\frac{F^n(a_n x+b_n)-\mathrm{e}^{-1/\tilde{y}}}{A(n)}+\mathrm{e}^{-1/\tilde{y}}\tilde{y}^{-\gamma-1}\bar{H}_{\gamma,\rho}(\tilde{y})\right| = o(1).$$

On the other hand, the assertion reads

$$\sup_{y\ge y_0} y^{1-\epsilon I_{\{\rho=0\}}}\left|\frac{F^n(a_nx+b_n)-e^{-1/y}}{A(n)}+e^{-1/y}y^{-\gamma-1}\tilde H_{\gamma,\rho}(y)\right|=o(1).$$

By (3.6) we have

$$\sup_{y\ge y_0}\frac{y^{1-\epsilon I_{\{\rho=0\}}}}{\tilde y^{1-\rho-\epsilon}}<\infty.$$

Hence, by Lemma 3.2, it suffices to prove that

$$\begin{aligned}\sup_{y\ge y_0} y^{1-\epsilon I_{\{\rho=0\}}}\Big|&\mathrm{e}^{-1/\tilde y}\tilde y^{-\gamma-1}\bar H_{\gamma,\rho}(\tilde y)\\ &-\mathrm{e}^{-1/y}y^{-\gamma-1}\tilde H_{\gamma,\rho}(y)+\frac{e^{-1/y}}{y}\cdot\frac{1}{\gamma+\rho}I_{\{\rho<0,\gamma+\rho\ne0\}}\Big|\longrightarrow 0. \qquad (3.7)\end{aligned}$$

Check that the term the absolute value of which is considered can be represented as $g(\tilde y)-g(y)$ with

$$g(t)=\begin{cases}\frac{1}{\gamma+\rho}t^{\rho-1}\mathrm{e}^{-1/t}, & \rho<0,\ \gamma+\rho\ne0,\\ t^{-\gamma-1}\log t\,\mathrm{e}^{-1/t}, & \rho<0,\ \gamma+\rho=0,\\ \frac{1}{\gamma}t^{-1}\log t\,\mathrm{e}^{-1/t}, & \rho=0\ne\gamma.\end{cases}$$

By the mean value theorem, the left-hand side of (3.7) equals

$$\sup_{y\ge y_0} y^{2-\epsilon I_{\{\rho=0\}}}\left|g'(\bar y)\Big(\frac{\tilde y}{y}-1\Big)\right|$$

for some $\bar y$ between y and $\tilde y$. Using (3.6), it is easily verified that

$$\sup_{y\ge y_0} y^{2-\epsilon I_{\{\rho=0\}}}|g'(\bar y)|=\sup_{\bar y\ge y_0}\bar y^{2-\epsilon I_{\{\rho=0\}}}|g'(\bar y)|(1+o(1))$$

is bounded. Hence the assertion follows readily from (3.6). □

Proof of Corollary 2.1: The first assertion follows from Theorem 2.1 and the boundedness of $(1+\gamma x)^{1/\gamma}(1-G_\gamma(x))$ for $x\ge x_0$. The second assertion is now obvious since

$$\frac{G_\gamma(x)}{1-G_\gamma(x)}(1+\gamma x)^{-1-1/\gamma}\tilde H_{\gamma,\rho}((1+\gamma x)^{1/\gamma})$$

is bounded for $x\ge x_0$. □

4 Appendix

In the present paper we use the second order condition (1.4) on $V=(1/(-\log F))^{\leftarrow}(t)$, while the analogous condition on $U=(1/(1-F))^{\leftarrow}(t)$ is more common in the literature. In this appendix, we will discuss the relationship between these two conditions. To this end, we first examine the effect of certain transformations on the so-called second order extended regular variation, that is, condition (1.4) in a slightly more abstract framework.

Proposition 4.1. *Suppose $g \in ERV^{(2)}(\gamma_1,\, \rho_1)$ with $\gamma_1 \in \mathbb{R}$, $\rho_1 \le 0$, i.e.*

$$\frac{\frac{g(tx)-g(t)}{a_1(t)} - \frac{x^{\gamma_1}-1}{\gamma_1}}{A_1(t)} \to H_{\gamma_1,\,\rho_1}(x) \tag{4.1}$$

and f satisfies

$$\frac{\frac{f(tx)}{f(t)} - x^{\gamma_2}}{A_2(t)} \to x^{\gamma_2}\frac{x^{\rho_2}-1}{\rho_2} \tag{4.2}$$

with $\gamma_2 > 0$ and $|A_2| \in RV(\rho_2)$ for some $\rho_2 \le 0$ and $A(t) \to 0$ as $t \to \infty$. If

$$\frac{A_1(f(t))}{A_2(t)} \to c \tag{4.3}$$

as $t \to \infty$ for some $c \in [-\infty,\, +\infty]$, then

$$\begin{aligned} &\frac{\frac{g(f(tx))-g(f(t))}{\gamma_2 a_1(f(t))} - \frac{x^{\gamma_1\gamma_2}-1}{\gamma_1\gamma_2}}{|A_1(f(t))| + |A_2(t)|} \\ &\to sgn(A_2)\left(\frac{1}{1+|c|}\cdot\frac{x^{\gamma_1\gamma_2}}{\gamma_2}\frac{x^{\rho_2}-1}{\rho_2} + \frac{c}{1+|c|}\cdot\frac{1}{\gamma_2}H_{\gamma_1,\rho_1}(x^{\gamma_2})\right) \end{aligned} \tag{4.4}$$

with $sgn(A_2)$ denoting the eventually constant sign of $A_2(t)$ and $c/(1+|c|)$ defined as ± 1 for $c \pm \infty$.

Corollary 4.1. *(i) Suppose $U \in ERV^{(2)}(\gamma, \rho)$ with $\gamma \in \mathbb{R}$, $\rho \le 0$ and auxiliary functions a and A. If $2tA(t) \to c \in [-\infty,\, +\infty] \setminus \{1-\gamma\}$, then*

$$\frac{\frac{V(tx)-V(t)}{a^*(t)} - \frac{x^{\gamma}-1}{\gamma}}{A^*(t)} \to H_{\gamma,\rho^*}(x) \tag{4.5}$$

as $t \to \infty$ for all $x \in \mathbb{R}$ with

$$\begin{aligned} \rho^* &= \max(\rho, -1), \\ a^*(t) &= \Big(1 - \frac{\gamma}{1+|c|}A_0(t)\Big)a\Big(\frac{1}{1-\mathrm{e}^{-1/t}}\Big), \\ A^*(t) &= \frac{\gamma-1+c}{1+|c|}A_0(t), \\ A_0(t) &= \Big|A\Big(\frac{1}{1-\mathrm{e}^{-1/t}}\Big)\Big| + \frac{1}{2t}. \end{aligned}$$

(ii) Conversely, suppose $V \in ERV^{(2)}(\gamma, \rho)$ with $\gamma \in \mathbb{R}$, $\rho \le 0$ and auxiliary functions a and A. If $2tA(t) \to c \in [-\infty,\, +\infty] \setminus \{\gamma-1\}$, then

$$\frac{\frac{U(tx)-U(t)}{a^*(t)} - \frac{x^{\gamma}-1}{\gamma}}{A^*(t)} \to H_{\gamma,\rho^*}(x)$$

as $t \to \infty$ for all $x \in \mathbb{R}$ with

$$\begin{aligned} \rho^* &= \max(\rho, -1), \\ a^*(t) &= \Big(1 + \frac{\gamma}{1+|c|} A_0(t)\Big) a\Big(\frac{1}{1-\mathrm{e}^{-1/t}}\Big), \\ A^*(t) &= \frac{1-\gamma+c}{1+|c|} A_0(t). \end{aligned}$$

Remark 4.1. *In the case $c = \gamma - 1$ V may or may not belong to $ERV^{(2)}(\gamma, \rho^*)$ for some $\rho^* \le 0$ if $U \in ERV^{(2)}(\gamma, \rho)$.*

Proof of Proposition 4.1: Because (4.2) with $\gamma_2 > 0$ implies $f(t) \to \infty$ as $t \to \infty$ and convergence (4.1) holds locally uniformly, one has for fixed $x > 0$

$$\begin{aligned} &\frac{g(f(tx)) - g(f(t))}{a_1(f(t))} \\ &= \frac{g(\frac{f(tx)}{f(t)} \cdot f(t)) - g(f(t))}{a_1(f(t))} \\ &= \frac{(\frac{f(tx)}{f(t)})^{\gamma_1} - 1}{\gamma_1} + A_1(f(t)) H_{\gamma_1,\rho_1}\left(\frac{f(tx)}{f(t)}\right) + o(A_1(f(t))) \\ &= x^{\gamma_1\gamma_2} \cdot \frac{(\frac{f(tx)}{x^{\gamma_2} f(t)})^{\gamma_1} - 1}{\gamma_1} + \frac{x^{\gamma_1\gamma_2} - 1}{\gamma_1} + A_1(f(t)) H_{\gamma_1,\rho_1}\left(\frac{f(tx)}{f(t)}\right) + o(A_1(f(t))). \end{aligned} \tag{4.6}$$

By (4.2), one has for fixed $x > 0$

$$\frac{f(tx)}{x^{\gamma_2} f(t)} = 1 + A_2(t) \frac{x^{\rho_2} - 1}{\rho_2} + o(A_2(t)).$$

Hence

$$\frac{(\frac{f(tx)}{x^{\gamma_2} f(t)})^{\gamma_1} - 1}{\gamma_1} = A_2(t) \frac{x^{\rho_2} - 1}{\rho_2} + o(A_2(t)). \tag{4.7}$$

From (4.6) and (4.7), one may conclude

$$\begin{aligned} &\frac{g(f(tx)) - g(f(t))}{\gamma_2 a_1(f(t))} - \frac{x^{\gamma_1\gamma_2} - 1}{\gamma_1\gamma_2} \\ &= \frac{x^{\gamma_1\gamma_2}}{\gamma_2} \left(A_2(t) \frac{x^{\rho_2} - 1}{\rho_2} + o(A_2(t))\right) + \frac{1}{\gamma_2} A_1(f(t)) H_{\gamma_1,\rho_1}(x^{\gamma_2}) + o(A_1(f(t))). \end{aligned} \tag{4.8}$$

By (4.3), we can easily get (4.4).

Proof of Corollary 4.1:
(i) The function f defined by $f(t) = 1/(1-e^{-1/t})$ satisfies (4.2) with $\gamma_2 = 1$, $\rho_2 = -1$,

and $A_2(t) = -1/2t$. Since $V(t) = U(f(t))$ and $A(f(t))/A_2(t) = -2tA(t)(1 + o(1))$, Proposition 4.1 yields

$$\frac{\frac{V(tx)-V(t)}{a(f(t))} - \frac{x^\gamma - 1}{\gamma}}{A_0(t)} \longrightarrow \frac{1}{1+|c|}(x^{\gamma-1} - x^\gamma) + \frac{c}{1+|c|} H_{\gamma,\rho}(x).$$

Hence, because of $1/(1 + y) = 1 - y + o(y)$ as $y \to 0$,

$$\begin{aligned}
&\frac{V(tx) - V(t)}{a^*(t)} \\
&= \frac{x^\gamma - 1}{\gamma}\Big(1 + \frac{\gamma}{1+|c|}A_0(t) + o(A_0(t))\Big) \\
&\quad + \Big(\frac{1}{1+|c|}(x^{\gamma-1} - x^\gamma) + \frac{c}{1+|c|}H_{\gamma,\rho}(x)\Big)A_0(t) + o(A_0(t)) \\
&= \frac{x^\gamma - 1}{\gamma} + \Big(\frac{1}{1+|c|}(x^{\gamma-1} - 1) + \frac{c}{1+|c|}H_{\gamma,\rho}(x)\Big)A_0(t) + o(A_0(t)). \qquad (4.9)
\end{aligned}$$

If $\gamma = 1$ then $x^{\gamma-1} - 1$ vanishes, and the assertion is obvious, because c is assumed unequal to $1 - \gamma = 0$.

If $\gamma \neq 1$, then $x^{\gamma-1} - 1 = (\gamma - 1)H_{\gamma,-1}(x)$. So if $|c| = \infty$ (which implies $\rho \geq -1$) or $c = 0$ (and hence $\rho \leq -1$), then (4.5) is immediate from (4.9). Finally, if $c \in \mathbb{R} \setminus \{0, 1 - \gamma\}$, then necessarily $\rho = -1$ and $\gamma - 1 + c \neq 0$, so that again the assertion follows from (4.9).

(i) The proof is very similar to the one of (i). Here we use $f(t) = 1/(-\log(1 - 1/t))$, satisfying (4.2) with $\gamma_2 = 1$, $\rho_2 = -1$, and $A_2(t) = 1/2t$.

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