Error Bounds in Low Rank Matrix Recovery

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A problem of estimation of a large $m \times m$ Hermitian matrix A based on i.i.d. measurements

$$Y_j = \mathrm{tr}(AX_j) + \xi_j, \ j = 1, \ldots, n,$$

where X_j are random $m \times m$ matrices and $\{\xi_j\}$ is a zero mean random noise will be considered. The goal is to estimate A in the case when it has relatively small rank and with an error proportional to the rank. This problem has been extensively studied in the recent years, especially, in the case of noiseless measurements (e.g., Candes and Recht (2009); Candes and Tao (2009)). There are many important instances of the problem including matrix completion, when a random sample of entries of A is observed, and quantum state tomography, when A is a density matrix (that is, a Hermitian nonnegatively definite matrix of unit trace) and the goal is to estimate A based on the measurements of observables $X_1, \ldots, X_n$. We will discuss an approach to such problems based on penalized least squares methods with complexity penalties defined either in terms of nuclear norm, or in terms of von Neumann entropy (in the case of estimation of a density matrix). In particular, we will discuss probabilistic inequalities showing the dependence of the estimation error on the rank of A and other parameters of the problem, such as the number n of measurements and the variance of the noise. The proofs of these inequalities rely on a variety of tools including concentration inequalities, generic chaining bounds and noncommutative extensions of classical exponential bounds for sums of independent random variables.