Eulerian edge sets in locally finite graphs

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Abstract

In a finite graph, an edge set Z is an element of the cycle space if and only if every vertex has even degree in Z. We extend this basic result to the topological cycle space, which allows infinite circuits, of locally finite graphs. In order to do so, it becomes necessary to attribute a parity to the ends of the graph.

1 Introduction

In a series of three papers [7, 8, 9], Diestel and Kühn introduced the *topological cycle space* that allows to extend theorems about circuits and the cycle space in finite graphs to an important class of infinite graphs, the class of locally finite graphs. (A graph is locally finite if each vertex has finite degree.)

Previously, the cycle space of an infinite graph was often defined in the same way as for finite graphs, namely as the set of (finite) mod 2 sum of circuits, the edge sets of 2-regular connected subgraphs. With this naive definition many results about circuits either become trivial are outright false in locally finite graphs. Diestel and Kühn, on the other hand, define a circuit to be the edge set of a homeomorphic image of the unit circle in the graph compactified by its ends (an end is an equivalence class of rays; for precise definitions see next section). This definition not only includes the traditional, finite circuits but also allows infinite ones. As an example consider Figure 1. There the (edge set of the) double ray D is a circuit, since both tails of D are in the same end to the left. On the other hand, double ray D' is not a circuit. Yet, the union of D' and D'' is a circuit.

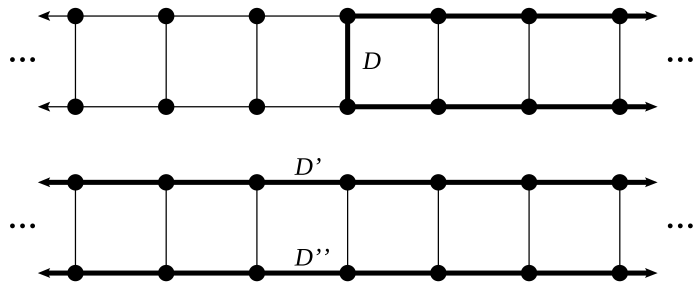

Figure 1: Circuits in the double ladder

The topological cycle space, which is defined to be the set of all (possibly infinite but well defined) mod 2 sums of circuits, permits verbatim extensions of many theorems to locally finite graphs. Among these are MacLane's planarity criterion [3], Tutte's generating theorem [1] and Gallai's partition theorem [2]. Moreover, since the circuits of Diestel and Kühn may be infinite, it becomes possible to consider Hamilton circuits in locally finite graphs, see for instance Georgakopoulos [10] or Cui, Wang and Yu [5].

However, one of the most basic and simple results characterising the cycle space of a finite graph did not so far have an analogue in locally finite graphs. For an edge set Z let us call a vertex Z-even if it is incident with an even number of edges in Z.

Proposition 1. *Let G be a finite graph. Then an edge set $Z \subseteq E(G)$ is an element of the cycle space if and only if every vertex in G is Z-even.*

Easy examples show that the proposition, as it is, cannot carry over to infinite graphs. All the vertices in the double rays D and D', for instance, have degree 2, yet $E(D)$ is a circuit but $E(D')$ is not.

The key difference between D and D' obviously lies in their behaviour at the ends of the double ladder. In [8] it was proposed to find a suitable definition for the *degree of an end* that captures this behaviour. Such a definition has been offered in [4], where the degree of an end is defined to be the maximal number of edge-disjoint rays in the end. Should this degree be finite, we call an end *even* if the end-degree is an even number, otherwise it is *odd*. For ends of infinite degree it is still possible to assign a parity, even or odd; we defer the details to Section 3. The concept of an end-degree allows to prove the following theorem, the main result of Bruhn and Stein [4].

Theorem 2. **[4]** *Let G be a locally finite graph. Then $E(G)$ is an element of the topological cycle space of G if and only if every vertex and every end of G has even degree.*

To see whether a subset Z of the egdes of a finite graph $G = (V, E)$ is an element of the cycle space, we evidently need to check the degree a given vertex has in the subgraph (V, Z), whereas the degree in the whole graph is irrelevant. In the same way, if we want to extend Proposition 1, we have to measure the degree of an end with respect to Z. Such an end-degree, that classifies ends as *Z-even* or as *Z-odd*, has been introduced in [4], which allowed to formulate the following conjecture.

Conjecture 3. **[4]** *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Then Z is an element of the topological cycle space of G if and only if every vertex and every end of G is Z-even.*

The purpose of this paper is to give a proof of the conjecture, which will be achieved over the course of Sections 4 and 5. In the next section we provide a formal definition of the topological cycle space, and in Section 3 we briefly discuss and define the notion of an end-degree.

2 Definitions

We refer the reader to Diestel [6] for general graph-theoretic definitions and notation. Let $G = (V, E)$ be a locally finite graph. We call a 1-way infinite path in G a *ray*, a 2-way infinite path is a *double ray*. Two rays are said to be *equivalent* if they cannot be separated by finitely many vertices, or equivalently, if there are infinitely many disjoint paths between them. The equivalence classes of the rays in G are called the *ends of* G.

We define a topological space, denoted by $|G|$, on the space consisting of G viewed as a 1-complex plus its ends. Thus, every edge is homeomorphic to the unit interval and basic open neighbourhoods of vertices consists of a choice of half-edges, one for each incident edge. For an end ω, each finite vertex set S defines a basic open neighbourhood as follows. Denote by $C(S, \omega)$ the one component of $G - S$ that contains a ray in ω and then a subray of every ray in ω. The basic open neighbourhood determined by S is the union of $C(S, \omega)$, all edges between S and $C(S, \omega)$ minus their endvertices in S, and all ends that have a ray in $C(S, \omega)$. The resulting topological space $|G|$, sometimes called the *Freudenthal compactification of* G, is compact and Hausdorff.

The homeomorphic image of the unit interval in $|G|$ is called an *arc*. For $S, T \subseteq V(G) \cup \Omega(G)$, we say that A is an S–T *arc* if the first point of A lies in S, the last in T and no interior point in $S \cup T$. For $x \in V(G) \cup \Omega(G)$ we simply speak of x–T *arcs* instead of $\{x\}$–T arcs, and proceed analogously for other combinations of singeltons and sets.

As defined by Diestel and Kühn [7] we say that a *circuit* of G is the edge set of a homeomorphic image of the unit circle in $|G|$; we remark that such a circle in $|G|$ contains every edge of which it contains an interior point.

We call a family $(F_i)_{i \in I}$ of subsets of E a *thin family* if every edge appears in at most finitely many of the F_i. The *sum* of such a family is denoted by $\sum_{i \in I} F_i$ and defined to be the edge set consisting of those edges that appear in exactly an odd number of the F_i. All sums of edge sets in this paper will be considered to be sums of thin families. The *topological cycle space* $\mathcal{C}(G)$ of G is the set of all sums of thin families of circuits. For more on the topological cycle space and on $|G|$ see [6]. For a more general concept of a cycle space in infinite graphs we defer the reader to Vella and Richter [12].

The following theorem gives a combinatorial characterisation of the cycle space.

Theorem 4 (Diestel and Kühn [7])**.** *Let Z be a set of edges in a locally finite graph G. Then $Z \in \mathcal{C}(G)$ if and only if Z meets every finite cut of G in an even number of edges.*

We call a subgraph R a *region* of G if there is a finite cut F of G so that R is a component of $G - F$. In particular, R is induced and connected.

Let $Z \subseteq E(G)$, and let C be some subgraph of G. We write $\partial_Z C$ for the edges in Z with exactly one endvertex in C and exactly one endvertex outside C. We write $\partial_G C$ for $\partial_{E(G)} C$.

For an edge set $Z \subseteq E$, we say that a vertex is *Z-even* if it is incident with an even number of edges in Z. We call a region R *Z-even* if $|\partial_Z R|$ is even, otherwise R is *Z-odd*.

3 End-degrees in subgraphs

In this section, let us first give the formal definition of the degree of an end with respect to the whole graph. In a second step we then refine the definition, so that it applies to subgraphs as well. We follow here the exposition of [4], where a more thorough discussion can be found.

Let ω be an end of a locally finite graph G. The *end-degree of* ω is then the supremum (in fact, this is a maximum) over the cardinalities of sets of edge-disjoint rays in ω, and we denote this (possibly infinite) number by $\deg(\omega)$.

For Theorem 2 the numerical value $\deg(\omega)$ is not important. Rather, it is essential whether ω can be said to be even or odd. Provided $\deg(\omega)$ is finite then it is obvious that ω should be even if and only if the number $\deg(\omega)$ is even. That raises the question what parity we should assign to an end ω of infinite degree. The graphs in Figure 2 demonstrate that we cannot call such an end always even or always odd. In both graphs all the vertices have even degree and all the ends have infinite degree. Yet, as can be easily checked with the help of Theorem 4, the edge set of the infinite grid lies in the cycle space, while for the graph H on the right, we have $E(H) \notin \mathcal{C}(H)$. Consequently, for Theorem 2 to become a true statement, the single end in the infinite grid has to be even, but the two ends in H should be odd.

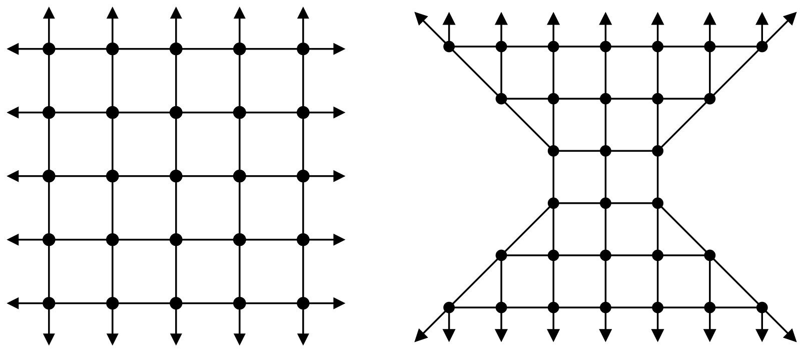

Figure 2: Even and odd ends of infinite degree

The example indicates that we need to distinguish between ends that have infinite degree but are even and ends of odd-infinite degree. This is accomplished by the following definition. We call an end ω *even* if there exists a finite vertex set $S \subseteq V(G)$ so that for all finite vertex sets $S' \supseteq S$ it holds that the maximal number of edge-disjoint rays in ω starting in S' is even. Otherwise, the end is called *odd*.

Let us make two remarks, both of which are discussed in more detail in [4]. First, for an end ω of finite degree, the end is even according to this definition if and only if $\deg(\omega)$ is an even number. Second, the choice of quantifiers for S and S' might appear arbitrary. Indeed, defining an end to be even with reversed quantifiers seems equally reasonable, i.e. an end ω would be even if *for all* finite vertex sets S *there exists* a finite superset S' such that the maximal number of edge-disjoint rays in ω starting in S' is even. While we suspect that such a definition of *weakly even* ends would still lead to Theorem 2 and Conjecture 3 to be true, we unfortunately cannot prove much in that respect.

Going back to Figure 2, we see that a choice of $S = \emptyset$ is sufficient for the single end of the infinite grid to be even. For any of the two ends of the graph on the right, however, it is not hard to check that as long as S' separates the two ends, the maximal number of edge-disjoint rays in the end starting in S' is odd, and hence the end itself is odd, as desired.

As outlined in the introduction, the degree of an end with respect to the whole graph is not of much use to us. To be able to decide whether a given edge set Z lies in the cycle space or not, we require a notion of an end-degree that takes Z into account. The key to adapting the notions introduced above, lies in substituting every occurrence of the word 'ray' by 'arc'. For this, denote by $\overline{Z}$ the closure of the point set $\bigcup_{z\in Z} z$ in $|G|$. We say that an end ω is *Z-even* if there exists a finite vertex set S so that for all finite vertex sets $S' \supseteq S$ it holds that the maximal number of edge-disjoint S'–ω arcs contained in $\overline{Z}$ is even. If an end is not Z-even, it is *Z-odd.* This definition is consistent with the definition of the end-degree in the whole graph, i.e. an end is $E(G)$-even if it is even Moreover, if the maximal number N of edge-disjoint arcs contained in $\overline{Z}$ and ending in ω is finite, then ω is Z-even if and only if N is even. For both these facts, see [4].

Consider the double rays D and D' in Figure 1 again. There are two (edge-disjoint) arcs contained in $\overline{E(D)}$ that end in the end to the right, so that that end is $E(D)$-even. So, all vertices and all ends are $E(D)$-even, and indeed $E(D)$ is a circuit. In contrast, $E(D')$ is not a circuit and we can easily check that any two arcs contained in $\overline{E(D')}$ terminating in the same end share an edge. Consequently, the ends are $E(D')$-odd and therefore a certificate for $E(D') \notin \mathcal{C}$.

Let us now state our main result.

Theorem 5. *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Then $Z \in \mathcal{C}(G)$ if and only if every vertex and every end of G is Z-even.*

The following lemma will be convenient when we check whether a given end is even or odd. We say that an edge set F *separates* a vertex set S from an end ω if every ray in ω with first vertex in S meets F.

Lemma 6. *Let ω be an end of a locally finite graph G, let $Z \subseteq E(G)$ and let $S \subseteq V(G)$ be a finite vertex set. Then the maximal number of edge-disjoint S–ω arcs contained in $\overline{Z}$ equals the minimum of $|F \cap Z|$ over all finite cuts F of G that separate S from ω.*

While the proof of the lemma is not overly difficult, it is not very instructive, and similar arguments have been given in [4]. We note that it can also be obtained from a more general result by Thomassen and Vella [11], who prove a Menger-type theorem for graph-like spaces.

With the help of Lemma 6 and Theorem 4 it becomes easy to prove the forward direction of our main theorem.

Lemma 7. *Let G be a locally finite graph, and let $Z \subseteq E(G)$. If $Z \in \mathcal{C}(G)$ then every vertex and every end of G is Z-even.*

Proof. If $Z \in \mathcal{C}(G)$ then, by definition or by Theorem 4, every vertex of G is Z-even. To see that any end ω is Z-even too, consider an arbitrary finite vertex set S'. By Lemma 6, the maximal number of edge-disjoint S'–ω arcs contained in $\overline{Z}$ equals the minimum $|F \cap Z|$ for all finite cuts F that separate S' from ω. Since Theorem 4 implies that for any finite cut F we have that $F \cap Z$ is an even set, we deduce that the number of S'–ω arcs is even, and consequently that ω is Z-even. □

The rest of the paper will be spent on proving the backward direction. For this, given an edge set Z in a locally finite graph G we assume that every vertex is Z-even but that $Z \notin \mathcal{C}(G)$. Hence, our aim is to find a Z-odd end, which we shall achieve be showing that the conditions of the next lemma are met.

Lemma 8. *Let G be a locally finite graph, and let Z be a subset of $E(G)$. Assume there exists a sequence $C_1, C_2, \dots$ of regions of G with the following properties:*

(i) $|\partial_Z C_n|$ *is odd for all n;*

(ii) $\partial_G C_n \cup E(C_n) \subseteq E(C_{n-1})$ *for all n; and*

(iii) *for every region R of G with $C_k \supseteq R \supseteq C_\ell$ for some $k \leq \ell$ it holds that $|\partial_Z R| \geq |\partial_Z C_k|$.*

Then, G contains a Z-odd end.

Proof. Denote by ω the end of a ray that meets every C_n (that such a ray exists can be seen, for instance, by Lemma 8.2.2 in [6]), and let us show that ω is Z-odd. First, observe that, by (ii), any arc that meets every C_n for large n contains a subarc with ω as endpoint. Now, let a finite vertex set S be given. By (ii), we may pick an N so that S is disjoint from C_N. Put $S' := S \cup N(G - C_N)$ and note that (iii) together with Lemma 6 imply that the maximal number of edge-disjoint S'–ω arcs contained in $\overline{Z}$ equals $|\partial_Z C_N|$, which is odd by (i). Thus, ω is Z-odd. □

Let us give a rough outline of the proof of Theorem 5. Lemma 8 provides us with a recipe for proving the existence of a Z-odd end. But how do we find Z-odd regions C_n as in the lemma? We first note that there is a natural candidate for C_1. By Theorem 4, there exists a finite cut of G that meets Z in an odd number of edges. Now, among all finite cuts F so that $|F \cap Z|$ is odd we choose one where $|Z \cap F|$ is minimal. Then, we take C_1 to be a component of $G - F$. This already ensures that for any Z-odd region $R \subseteq C_1$ it holds that $|\partial_Z R| \geq |\partial_Z C_1|$. Furthermore, since every vertex is Z-even the Z-odd cut $F = \partial_G C_1$ propagates into C_1, in the sense that C_1 properly contains Z-odd regions. Finding a suitable region C_2 is more difficult. In a similar way as for C_1, it appears enticing to simply pick among all Z-odd regions C with $\partial_G C \cup E(C) \subseteq E(C_1)$ one so that $|\partial_Z C|$ is minimal. However, since we chose $|\partial_Z C_1|$ to be minimal among all Z-*odd* regions there still could exists a Z-even region R sandwiched between C_1 and C_2 with smaller cut-size in Z than C_1, i.e. with $|\partial_Z C_1| > |\partial_Z R|$.

In order to overcome this problem, we will eliminate all small Z-even cuts before choosing C_2. This will be achieve by contracting certain Z-even regions and obtaining a minor all of whose infinite Z-even regions have large cut-size. In that minor we then choose a region C_2^* so that $\partial_Z C_2^*$ has minimal odd size. By uncontracting we obtain the region C_2 in the original graph. We repeat this procedure. Once again we eliminate all small Z-even cuts, choose C_3^* in the resulting minor and so on. This way we can obtain regions $C_1^*, C_2^*, \dots$ of different minors of G. We will gain the regions C_n of G by uncontracting the regions C_n^*.

The next section will hand us a tool to eliminate infinite Z-even regions of small cut-size. The main work of constructing a sequence of regions $C_1 \supseteq C_2 \supseteq \dots$ will be achieved in Section 5.

4 Elimination of regions with small cutsize

Before we can prove Lemma 11, the main tool to eliminate small Z-even cuts, we state two lemmas. The first of which is a standard lemma, that asserts that the function measuring the number of edges leaving a vertex set is submodular.

Lemma 9. *Let G be a graph, and let $X, Y \subseteq V(G)$. Then*

$$|\partial X| + |\partial Y| \geq |\partial(X \cap Y)| + |\partial(X \cup Y)|$$

and

$$|\partial X| + |\partial Y| \geq |\partial(X - Y)| + |\partial(Y - X)|.$$

The next lemma is used in the inductive proof of Lemma 11. For an edge set Z, we say that D is a *(m, Z)-region* if D is a region with $|\partial_Z D| = m$.

Lemma 10. *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Let C be a region of G, and denote by m the minimal integer k for which there is an infinite (k, Z)-region R in G with $R \subseteq C$. Assume m to be even. Let $R, S_1, \ldots, S_\ell$ be (m, Z)-regions, where $S_1, \ldots, S_\ell$ are pairwise disjoint and $|R - \bigcup_{i=1}^{\ell} S_i| = \infty$ Then there exist a subgraph K and an (m, Z)-region S satisfying*

(i) *the subgraph K is the union of components of $R - \bigcup_{i=1}^{\ell} S_i$;*

(ii) *the region S is spanned by the union of K with some (possibly none) of $S_1, \ldots, S_\ell$;*

(iii) *$S - S_i$ is connected for every $i = 1, \ldots, \ell$;*

(iv) *K is an infinite subgraph and $|\partial_Z K|$ is even; and*

(v) *if $m = 0$ then each K is connected; and if $m > 0$ then each component of K is incident with an edge in Z.*

Proof. Define $\mathcal{I}$ to be the set of those S_i among $S_1, \ldots, S_\ell$ for which $S_i - R$ is infinite; denote by $\mathcal{J}$ the other ones. Consider an $S_i \in \mathcal{I}$. Observe that by definition of m and since each of $R - S_i$ and $S_i - R$ is an infinite subgraph, Lemma 9 implies that $|\partial_Z(R - S_i)| = |\partial_Z(S_i - R)| = m$. Hence, $R - S_i$ contains an infinite (m, Z)-region. In a similar way, we see that for any $S_j \in \mathcal{J}$, the induced subgraph on $R \cup S_j$ contains an infinite (m, Z)-region. As the $S_1, \ldots, S_\ell$ are pairwise disjoint it follows therefore that each infinite component of $G[(R - \bigcup \mathcal{I}) \cup \bigcup \mathcal{J}]$ (and there is at least one) is is an infinite (m, Z)-region. For one of these components, R' say, the subgraph $K' := R' - \bigcup_{i=1}^{\ell} S_i$ will be infinite, so that K' satisfies (i), and (ii) holds for R'. Among all infinite subgraphs K of G and (m, Z)-region S satisfying (i) and (ii), we choose S to be $\subseteq$-minimal.

Let us now show that K satisfies (iv). Indeed, let $\mathcal{T} \subseteq \{S_1, \ldots, S_\ell\}$ so that $S = G[K \cup \bigcup \mathcal{T}]$. Since the S_i are pairwise disjoint it follows that

$$\partial_Z K = \partial_Z(S - \bigcup \mathcal{T}) = \partial_Z S + \sum_{T \in \mathcal{T}} \partial_Z T.$$

(Recall that we consider the sum of edge sets to be their symmetric difference.) Since S as well as all the $T \in \mathcal{T}$ are (m, Z)-regions, we deduce that $|\partial_Z K|$ is even.

Next, assume that $m > 0$ and suppose that K has a component L that is not incident with any edge in Z. Since $m > 0$, L cannot be infinite, which implies that $K - L$ is still infinite. Moreover, as $|\partial_Z(S - L)| = m$, one of the components of $S - L$ is an infinite (m, Z)-region S', which then together with $K' := S' - \bigcup_{i=1}^{\ell} S_i$ constitutes a contradiction to the minimal choice of S. If, on the other hand, $m = 0$ then K cannot be disconnected as each infinite component K' of K would with $S' = K'$ contradict the choice of K and S. Therefore, (v) is proved.

Finally, in order to prove (iii), suppose that there exists a k so that $S - S_k$ is not connected. Since $K \subseteq S - \bigcup_{i=1}^{\ell} S_i$ is infinite one of the components of $S - S_k$, X say, has therefore the property that $K' := X - \bigcup_{i=1}^{\ell} S_i$ is infinite as well. Observe that it follows from (ii) that $S_k \subseteq S$. Setting $Y := S - (S_k \cup X)$, we see that

$$2m = |\partial_Z S| + |\partial_Z S_k| = |\partial_Z(S_k \cup X)| + |\partial_Z(S_k \cup Y)|.$$

As both $S_k \cup X$ and $S_k \cup Y$ are infinite, $S' := G[S_k \cup X] \subseteq S$ is an infinite (m, Z)-region. Again we have, with S' and K', obtained a contradiction to the minimal choice of S. $\square$

In the rough sketch of the proof of Theorem 5, we claimed we would construct minors in order to eliminate infinite Z-even regions of small cutsize. This is slightly incorrect. Unfortunately, and this will lead to some technical complications, we are not able to force the contracted branch sets to be connected. Rather, it will sometimes be necessary to contract a disconnected set to a single vertex. Thus, we will not be working with minors but with what we call pseudo-minors.

Let $\mathcal{V}$ be a partition of the vertex set of a graph G. We define a graph H with vertex set $\mathcal{V}$ and edge set $E(H) \subseteq E(G)$, so that e is an edge of H between two distinct vertices U and U' of H if and only if e is an edge of G with one endvertex in U and the other in U'. In particular, we allow H to have parallel edges but no loops. We call such a graph H a *pseudo-minor of* G, denoted by $H \preccurlyeq G$, and define $\mathcal{K}(H, G)$ to be the set of non-singletons in $\mathcal{V}$.

Let D, K be subgraphs of G. We say that D *splits* K if neither $V(K) \subseteq V(D)$ nor $V(D) \cap V(K) = \emptyset$.

Lemma 11. *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Let C be a region of G, and denote by m the minimal k for which there is an infinite (k, Z)-region R in G with $R \subseteq C$. Assume m to be even. Then there exists a locally finite pseudo-minor G' of G and a set $\mathcal{S}$ of (m, Z)-regions of G so that the following holds:*

(i) *K is infinite for each $K \in \mathcal{K}(G', G)$ and $|\partial_Z K|$ is even;*

(ii) *every region D of G splits at most finitely many $K \in \mathcal{K}(G', G)$;*

(iii) *if D is an infinite (k, Z)-region of G' with $E(D) \subseteq E(C)$ then it follows that $k > m$;*

(iv) *for every $K \in \mathcal{K}(G', G)$ there is an $S \in \mathcal{S}$ with $K \subseteq S \subseteq C$;*

(v) *for every $S \in \mathcal{S}$ there is an $\mathcal{L} \subseteq \mathcal{K}(G', G)$ with $S = G[\bigcup_{L \in \mathcal{L}} L]$; and*

(vi) *if* $m = 0$ *then each* $K \in \mathcal{K}(G', G)$ *is connected; and if* $m > 0$ *then each component of* K, $K \in \mathcal{K}(G', G)$, *is incident with an edge in* Z.

If the assertions of the lemma are satisfied for a graph G with pseudo-minor G', a region C and a set of regions $\mathcal{S}$ then we call the tuple $(G', G, C, \mathcal{S}, m)$ a *legal contraction system.*

Proof. We may restrict ourselves to the component of G that contains the region C, and therefore assume that G itself is connected. Let $R_1, R_2, \ldots$ be an enumeration of all infinite (m, Z)-regions of G. (Since G is connected, these are only countably many.)

We shall define inductively subgraphs $K_1, K_2, K_3, \ldots$ and (m, Z)-regions $S_1, S_2, S_3, \ldots \subseteq C$ satisfying

- K_i is infinite for each $i = 1, 2, 3, \ldots$ and $|\partial_Z K_i|$ is even;
- For every $i = 1, 2, 3, \ldots$, the region S_i is spanned by the union of K_i with some (possibly none) of $S_1, \ldots, S_{i-1}$;
- The $K_1, K_2, K_3, \ldots$ are pairwise disjoint;
- if $m = 0$ then each K_i is connected; and if $m > 0$ then each component of each K_i is incident with an edge in Z.

We note that the second and third property imply that

$$\textit{for each } j < i \textit{ either } S_j \subseteq S_i \textit{ or } S_j \cap S_i = \emptyset. \tag{1}$$

Taking $\mathcal{S} = \{S_1, S_2, S_3, \ldots\}$ and obtaining G' from G by contracting the K_i, we clearly have (i), (iv)–(vi). A further analysis of the process will yield (ii) and (iii).

We start by setting $K_1 = S_1 = R_1$. Now assume that $K_1, \ldots K_\ell$ and $S_1, \ldots, S_\ell$ are constructed. We denote by n_ℓ the minimal n satisfying $|R_n - \bigcup_{i=1}^{\ell} S_i| = \infty$. (If no such R_n exists, the process terminates.) We then apply Lemma 10 to R_{n_ℓ} and the $\subseteq$-maximal regions among $S_1, \ldots, S_\ell$, which are, by (1), pairwise disjoint. The resulting K and S found by the lemma will be chosen as $K_{\ell+1}$ and $S_{\ell+1}$ respectively.

In order to see that (ii) is satisfied, let R be a region of G and let $X = (\partial_G R) \cap \bigcup_{i=1}^{\infty} E(S_i)$. Let $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$ be a set of minimal size with $X \subseteq \bigcup_{j=1}^{k} E(S_{i_j})$. We claim that R does not split any K_i with $i > \max(i_1, \ldots, i_k)$. To reach a contradiction, suppose that R splits K_n for some $n > \max(i_1, \ldots, i_k)$. Then, as K_n is disjoint from all the S_{i_j}, K_n must be disconnected, and have at least one component inside R and at least one component outside R. Since $K_n \subseteq S_n$ and since S_n is connected, S_n must contain some edge from X. Hence it meets some S_{i_j}, S_{i_1} say. As $n > i_1$, we have, by construction, that $S_{i_1} \subseteq S_n$.

Let S_p be the $\subseteq$-maximal region among $S_1, \ldots S_{n-1}$ containing S_{i_1}. Recall that we chose S_n and K_n using Lemma 10, which states that $S_n - S_p$ is connected. As, furthermore, K_n is disjoint from S_p because it is chosen later, it follows that $S_n - S_p$ contains an edge of X, and thus meets and then contains one of $S_{i_2}, \ldots, S_{i_k}$, say S_{i_2}. We thus have $S_{i_2} \subseteq S_n - S_p \subseteq S_n - S_{i_1}$. This, however, leads to $X \subseteq E(S_n) \cup \bigcup_{j=3}^{k} E(S_{i_j})$, which contradicts the minimality of k. This completes the proof of (ii).

Let us finally prove (iii). Note that if $n_{\ell+1} = n_\ell$ then, by Lemma 10 (i), $R_{n_\ell} - \bigcup_{i=1}^{\ell+1} S_i$ will have less components than $R_{n_\ell} - \bigcup_{i=1}^{\ell} S_i$. This implies $\lim_{\ell\to\infty} n_\ell = \infty$. Therefore, for every (m, Z)-region R_n it holds that $|R_n - \bigcup_{i=1}^{\ell} S_i| < \infty$ for some ℓ. Now, if D is an infinite (k, Z)-region of G' with $E(D) \subseteq E(C)$ then by uncontracting and (ii) we find a (k', Z)-region R of G with $R \subseteq C$ and $k' \leq k$ so that $E(R) \cap E(G')$ is infinite. By assumption, we have that $k' \geq m$. If $k' = m$ then $R = R_n$ for some n, and consequently $|R_n - \bigcup_{i=1}^{\ell} S_i| < \infty$ for some ℓ, contradicting that $E(R) \cap E(G')$ is infinite. This proves (iii). $\square$

5 Proof of main result

We restate and then prove our main result.

Theorem 5. *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Then $Z \in \mathcal{C}(G)$ if and only if every vertex and every end of G is Z-even.*

Proof. In light of Lemma 7 we only need to prove the backward direction. In order to do so, assume that every vertex of G is Z-even but that $Z \notin \mathcal{C}(G)$. Our task is to find a Z-odd end in G.

Since $Z \notin \mathcal{C}(G)$ there exists a topologically connected component of $\overline{Z}$ whose edge set is not an element of the cycle space (recall that $\overline{Z}$ is the closure of Z as a subspace of $|G|$). An end of G that is odd with respect to some connected component of $\overline{Z}$ is Z-odd, as each end of G lies in at most one connected component. Thus, we may, by deleting all other edges from Z, assume that $\overline{Z}$ is topologically connected. In particular, this means that

there exists no finite cut of G that avoids Z but separates two edges in Z. (2)

Recall that in order to apply Lemma 8, we need to find a sequence of nested Z-odd regions $C_1 \supsetneq C_2 \supsetneq \ldots$ of G so that for any region R of G with $C_k \supseteq R \supseteq C_\ell$ for some $k \leq \ell$ it holds that $|\partial_Z C_k| \leq |\partial_Z R|$. We shall do this by first defining a sequence of regions $C_1^*, C_2^*, \ldots$ in certain pseudo-minors of G. We will obtain C_n from the C_n^* by uncontracting.

Let us be more precise. Inductively, we will find Z-odd regions $C_1^*, C_2^* \ldots$ in certain pseudo-minors $G^0 \succcurlyeq G^2 \succcurlyeq \ldots$ of G. We set $m_n := |\partial_Z C_n^*| - 1$ for all $n \geq 1$, and for convenience we put $m_0 = -2$. We start the construction with $G^{-2} = G$. Now, in each step, i.e. for each $n \in \mathbb{N}$, we will first find a region C_n^* of $G^{m_{n-1}}$ and then construct pseudo-minors $G^{m_{n-1}+2} \succcurlyeq \ldots \succcurlyeq G^{m_n}$ of $G^{m_{n-1}}$. We require that for all $n \geq 1$ it holds that

(i) $|\partial_Z C_n^*|$ is odd; and

(ii) $\partial_{G^{m_{n-1}}} C_n^* \cup E(C_n^*) \subseteq E(C_{n-1}^*)$ if $n \geq 2$.

Let us pause for a while before we give a third requirement. Recall that C_n^* is a region in the pseudo-minor $G^{m_{n-1}}$. It should be noted that there might be values of m which are not equal to any m_n, but we will still need to refer to a region of G^m which is naturally obtained from the sequence $C_1^*, C_2^*, C_3^*, \ldots$. For this end, we introduce the following notation:

Let $\ell > m \geq -2$ be two even numbers and let D be an induced subgraph of G^m. Then $\pi_{m,\ell}(D)$ denotes the induced subgraph of G^ℓ on the vertex set $\{X_v : v \in V(D) \text{ and } v \in X_v \in V(G^\ell)\}$ (recall that, as G^ℓ is a pseudo-minor of G^m, its vertex set is a partition of $V(G^m)$). We will also consider the inverse of $\pi_{m,\ell}$, which we denote by $\pi_{\ell,m}$. Given an induced subgraph D' of G^ℓ define $\pi_{\ell,m}(D')$ to be the induced subgraph of G^m on the vertex set $\bigcup_{X\in V(D')} X$. Also for every induced subgraph D of G^m write $\pi_{m,m}(D) = D$. Thus $\pi_{m,\ell}(D)$ is defined for every $m, \ell \in \{-2, 0, 2, 4, \ldots\}$ regardless of the order between them, and for every induced subgraph D of G^m it assigns an induced subgraph of G^ℓ.

With this notation, put $C_n^m := \pi_{m_{n-1},m}(C_n^*)$ for every $m = -2, 0, 2, 4, \ldots$ and $n = 1, 2, 3, \ldots$. In particular, this means that $C_n^* = C_n^{m_{n-1}}$.

We are now ready to state the third requirement. Alongside with the construction of the pseudo-minors G^m and the regions C_n^* we will construct, for every m, sets $\mathcal{S}_m$ of (m, Z)-regions of G^{m-2} so that

(iii) $(G^m, G^{m-2}, C_n^{m-2}, \mathcal{S}_m, m)$ is a legal contraction system, where n is the number satisfying $m_{n-1} < m \leq m_n$.

Thus, by Lemma 11 (i) each vertex in G^m remains Z-even. Since it is central to the main idea of the proof we restate another implications of (iii):

if $D \subseteq C_n^m$ is an infinite (k, Z)-region of G^m for $0 \leq m \leq m_n$ then $k > m$. (3)

The statement follows from Lemma 11 (iii) either directly or by induction, depending on whether $m_{n-1} < m \leq m_n$ or not.

Furthermore, we note the following consequence of (iii) or, more specifically, of Lemma 11 (iv).

C_n^* does not split any $K \in \mathcal{K}(G^m, G^{m_{n-1}})$ for any even $m \geq m_{n-1}$. (4)

In particular, it follows that the cut $\partial_{G^{m_{n-1}}} C_n^*$ lies in G^m for each even $m \geq m_{n-1}$ and thus is still a cut there.

Let us make one last observation before we finally start with the construction. We claim that

if D' is an infinite region of G^m for some m then there exists for each even $\ell \leq m$ an infinite region D of G^ℓ with $\partial_{G^\ell} D \subseteq \partial_{G^m} D'$ and $\pi_{\ell,m}(D) \subseteq D'$. (5)

Indeed, $\tilde{D} := \pi_{m,\ell}(D')$ is an induced subgraph of G^ℓ with $\partial_{G^\ell}\tilde{D} = \partial_{G^m} D'$. From (iii), resp. Lemma 11 (ii), it follows that $\tilde{D}$ has only finitely many components. Hence, one of them is infinite, and this will be the desired region D.

We are now ready to start the construction. We begin with $G^{-2} = G$ and let $C_1 = C_1^*$ be a Z-odd region with the minimal possible value of $|\partial_Z C_1|$. We know that Z-odd regions exist by Theorem 4. Recall that we write $m_1 = |\partial_Z C_1| - 1$ We then construct $G^0, \ldots, G^{m_1}$ and $\mathcal{S}_0, \ldots, \mathcal{S}_{m_1}$ using Lemma 11 in a way that will be described in more details later on.

For $n > 1$, assume $C_1^*, \ldots, C_{n-1}^*$ and $G^{-2}, G^0, \ldots, G^{m_{n-1}}$ and the corresponding $\mathcal{S}_m$ to be constructed. In order to find a suitable region C_n^*, we first claim that there exist Z-odd regions C satisfying (ii).

For $n > 1$, denote by F the edges in $G^{m_{n-1}}$ incident with the vertices in $N(G^{m_{n-1}} - C_{n-1}^{m_{n-1}})$. Since C_{n-1}^* is a region, F is a finite set and thus $|F \cap Z|$

is even. Furthermore, $\partial_{G^{m_{n-1}}} C^*_{n-1} \subseteq F$ implies that the cut $F \setminus \partial_{G^{m_{n-1}}} C^*_{n-1}$ meets Z in an odd number of edges. One of the components of $G^{m_{n-1}} - (F \setminus \partial_{G^{m_{n-1}}} C^*_{n-1})$ contained in $C^{m_{n-1}}_{n-1}$ will thus be Z-odd and hence as desired. We now pick C^*_n among all Z-odd regions C of $G^{m_{n-1}}$ satisfying (ii) so that $|\partial_Z C^*_n|$ is minimal.

A consequence of the choice of C^*_n is that for all $n \in \mathbb{N}$:

if D is a (k, Z)-region of $G^{m_{n-1}}$ so that k is odd and $D \subseteq C^*_n$ then $k \geq m_n + 1 = |\partial_Z C^*_n|$. (6)

In order to define G^m, for even m with $m_{n-1} < m \leq m_n$, assume G^ℓ for $\ell = -2, 0, 2, 4, \ldots, m-2$ to be already constructed. By (3), it holds that the smallest k^* for which there is an infinite (k^*, Z)-region R of G^{m-2} with $R \subseteq C^{m-2}_n$ is at least $m-1$. Now, (6) in conjunction with (5) shows that $k^* \geq m$. Hence, we may apply Lemma 11 to G^{m-2}, Z, C^{m-2}_n, m (in the roles of G, Z, C, m respectively). With the resulting pseudo-minor G^m and set of (m, Z)-regions $\mathcal{S}_m$, the tuple $(G^m, G^{m-2}, C^{m-2}_n, \mathcal{S}_m, m)$ is a legal contraction system, i.e. (iii) is satisfied.

Assume the construction achieved for all n and corresponding m. Set $C_n := \pi_{m_{n-1},-2}(C^*_n) = C^{-2}_n$, i.e. C_n is the induced subgraph of G obtained from C^*_n by uncontracting $\mathcal{K}(G^{m_{n-1}}, G)$. Observe that

$|\partial_Z C_n|$ is odd and $\partial_G(C_n) \cup E(C_n) \subseteq E(C_{n-1})$ for all $n \in \mathbb{N}$. (7)

Recall that our aim is to find a sequence of regions C_n of G satisfying the requirements of Lemma 8. As (7) means that already two of the conditions hold we need only make sure that each C_n is indeed a region, i.e. a connected subgraph, and that for all regions R with $C_k \supseteq R \supseteq C_\ell$ for some $k \leq \ell$ it holds that $|\partial_Z R| \geq |\partial_Z C_k|$. We shall deal with the latter condition first.

Observe that (4) implies that

C_n does not split any $K \in \mathcal{K}(G^{m_n}, G)$. (8)

Next, we prove that for every n and $0 \leq m \leq m_n$ it holds that

for every (k, Z)-region R of G with $R \subseteq C_n$ and $k \leq m$ it follows that $\pi_{-2,m}(R)$ is a finite subgraph of G^m. (9)

Assume the statement to be false for G^m, for some m. For even ℓ, $-2 \leq \ell \leq m$, denote by $\mathcal{D}^\ell$ the set of (k, Z)-regions D of G^ℓ with $k \leq m$, $D \subseteq C^\ell_n$ and so that $\pi_{\ell,m}(D)$ is infinite. Clearly, by assumption we have that $\mathcal{D}^{-2} \neq \emptyset$. On the other hand, it holds that $\mathcal{D}^m = \emptyset$. Indeed, any element in $\mathcal{D}^m$ would contradict (3).

Now, choose $\ell \leq m$ to be the maximal even integer so that $\mathcal{D}^{\ell-2} \neq \emptyset$, and among the $D \in \mathcal{D}^{\ell-2}$ pick one, $\tilde{D}$ say, so that $\tilde{D}$ splits a minimum number of elements in $\mathcal{K}(G^\ell, G^{\ell-2})$. (Note that, by Lemma 11 (ii), every region splits only finitely many sets in $\mathcal{K}(G^\ell, G^{\ell-2})$.)

Now, since $\mathcal{D}^\ell = \emptyset$, the region $\tilde{D}$ of $G^{\ell-2}$ must split some $K \in \mathcal{K}(G^\ell, G^{\ell-2})$. Let $S \in \mathcal{S}_\ell$ be an (ℓ, Z)-region with $K \subseteq S$ of $G^{\ell-2}$ (see Lemma 11 (iv)).

We distinguish two cases. Assume that $S - \tilde{D}$ is infinite. If $|\partial_Z(\tilde{D} - S)| > |\partial_Z \tilde{D}|$ then, by Lemma 9, $S - \tilde{D}$ contains an infinite (ℓ', Z)-region of $G^{\ell-2}$ with

$\ell' < \ell$, in contradiction to either (3) or (6) (together with (5)). Thus, $\tilde{D} - S$ is a (k', Z)-region with $k' \le |\partial_Z \tilde{D}| \le m$. Observe that because $\pi_{\ell-2,\ell}(S)$ is finite by Lemma 11 (iii), the subgraph $\pi_{\ell-2,m}(\tilde{D} - S)$ of G^m is still infinite. As, moreover, by Lemma 11 (v), $\tilde{D} - S$ splits fewer elements in $\mathcal{K}(G^\ell, G^{\ell-2})$, we obtain a contradiction to the choice of $\tilde{D}$.

So, let $S - \tilde{D}$ be finite. Suppose that $|\partial_Z(S \cup \tilde{D})| > |\partial_Z \tilde{D}|$. Then, by Lemma 9, $S \cap \tilde{D}$ contains an infinite (ℓ', Z)-region of $G^{\ell-2}$ with $\ell' < \ell$, in contradiction to either (3) or (6). Thus, $G[S \cup \tilde{D}]$ is an infinite (k', Z)-region with $k' \le |\partial_Z \tilde{D}| \le m$. Since $G[S \cup \tilde{D}]$ splits fewer $K \in \mathcal{K}(G^\ell, G^{\ell-2})$ than $\tilde{D}$, we obtain again a contradiction to the choice of $\tilde{D}$—provided we can show that $S \cup \tilde{D} \subseteq C_n^{\ell-2}$. To do this, observe that, by Lemma 11 (v), there is a set $\mathcal{L} \subseteq \mathcal{K}(G^\ell, G^{\ell-2})$ with $S = G[\bigcup_{L \in \mathcal{L}} L]$. Since $S - \tilde{D}$ is finite but all the $L \in \mathcal{L}$ are infinite by Lemma 11 (i), it follows that $\tilde{D}$ meets every $L \in \mathcal{L}$. Together with the fact that C_n does not split any elements in $\mathcal{K}(G^\ell, G^{\ell-2})$, by (8), and $\tilde{D} \subseteq C_n^{\ell-2}$ it follows that $S \subseteq C_n^{\ell-2}$, and hence, $S \cup \tilde{D} \subseteq C_n^{\ell-2}$. This finishes the proof of (9).

In order to prove that the subgraphs $C_1, C_2, \ldots$ of G satisfy Condition (iii) of Lemma 8 consider a region R with $C_k \supseteq R \supseteq C_\ell$ for some $\ell \ge k$. Observe that $C_\ell^{m_k}$ is still an infinite subgraph of G^{m_k} since $\partial_Z C_\ell^{m_k}$ is odd but every vertex in G^{m_k} is Z-even. Thus, $\pi_{-2,m_k}(R) \supseteq C_\ell^{m_k}$ is infinite, which with (9) implies that $|\partial_Z R| \ge m_k + 1 = |\partial_Z C_k|$, as desired.

For Lemma 8 to apply, it remains to show that:

$$C_n \textit{ is a region of } G \textit{ for all } n \in \mathbb{N}. \tag{10}$$

For this, it suffices to prove that C_n is connected. If $m_n = 0$ then, by Lemma 11 (vi), G^0 is a minor (rather than only a pseudo-minor) of G. As C_n^* is a region in $G^{m_{n-1}}$, which is either G^0 or $G^{-2} = G$, we immediately see that C_n is connected as well.

So, let $m_n > 0$. Since C_n does not split any $K \in \mathcal{K}(G^{m_n}, G)$ and since $\partial_G C_n$ is odd, $C_n^{m_n}$ is infinite. Let C be a component of C_n so that $\pi_{-2,m_n}(C)$ is infinite, and suppose that $R := C_n - C$ is non-empty. If $\partial_Z R \ne \emptyset$ then $|\partial_Z C| \le m_n$ in contradiction to (9). If, on the other hand, $\partial_Z R = \emptyset$ then $\partial_G R$ is a finite cut of G separating two edges in Z, which constitutes a contradiction to that $\overline{Z}$ is topologically connected. Indeed, C contains an edge of Z since $\partial_Z C = \partial_Z C_n$ is an odd set but every vertex is Z-even. To see that $E(R)$ meets Z, recall that C_n^* is a region of $G^{m_{n-1}}$. Thus, there exists an $\ell \le m_{n-1}$, so that $\pi_{-2,\ell-2}(C)$ splits a $K \in \mathcal{K}(G^\ell, G^{\ell-2})$. By (8), the subgraph K of $G^{\ell-2}$ is contained in $C_n^{\ell-2}$ and then K has one component in $\pi_{-2,\ell-2}(C)$ and one in $\pi_{-2,\ell-2}(R)$. Lemma 11 (vi) implies that both these components are incident with an edge in Z. As $\partial_Z R = \emptyset$, we obtain $E(R) \cap Z \ne \emptyset$.

In conclusion, the regions C_n satisfy all conditions required in Lemma 8, which therefore yields the desired Z-odd end in G. $\square$

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