Hamilton circles in planar locally finite graphs

Henning Bruhn Xingxing Yu

Abstract

A classical theorem by Tutte assures the existence of a Hamilton cycle in every finite 4-connected planar graph. Extensions of this result to infinite graphs require a suitable concept of an infinite cycle. Recently, such a concept was provided by Diestel and Kühn, who defined circles to be as homeomorphic images of the unit circle in the Freudenthal compactification of the (locally finite) graph. Using this definition, which allows for infinite cycles, we present a partial extension of Tutte's result to locally finite graphs.

1 Introduction

While Hamilton cycles have been investigated intensively in finite graphs, comparatively little attention has been paid to Hamilton cycles in infinite graphs. One reason for this is that it is not entirely clear what the infinite analogon of a cycle should be.

Adomaitis [1] avoided this question by defining a graph to be hamiltonian if for every finite subset of the vertex set there is a spanning cycle. In contrast, Nash-Williams [7] addressed the problem and proposed spanning double rays as infinite analogons of Hamilton cycles. He noticed that for a spanning double ray to exist the graph needs to be 3-indivisible. (A graph is k-indivisible if the deletion of finitely many vertices leaves at most $k-1$ infinite components.) Generalising the following classical result by Tutte, Nash-Williams conjectured that a 3-indivisible 4-connected planar graph contains a spanning double ray.

Theorem 1 (Tutte [10]). *Every finite* 4*-connected planar graph has a Hamilton cycle.*

Recently, Yu [14, 15, 11, 12, 13] announced a proof of Nash-Williams' conjecture.

The restriction to 3-indivisible graphs is a quite serious one that at first appears unavoidable. Yet, while double rays are the obvious first choice for an infinite analogon of cycles there is a more subtle alternative, which was introduced by Diestel and Kühn [5, 6]. They call the homeomorphic image of the unit circle in the Freudenthal compactification of a locally finite graph a *circle*. In a series of papers it has been shown that this notion is very successful and more suitable than double rays; see Diestel [3] for an introduction and a survey.

These circles overcome the restriction to 3-indivisible graphs. Indeed, in Figure 1 we see a *Hamilton circle*, ie. a spanning circle, in a graph that is not k-indivisible for any k. The example is due to Diestel and Kühn [5]. Bruhn

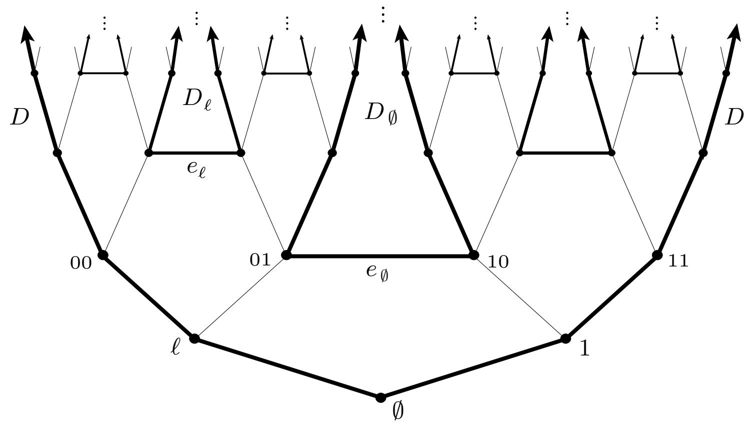

Figure 1: A Hamilton circle (drawn bold)

(see [3]) conjectured that, in this sense, Theorem 1 extends to locally finite graphs:

Conjecture 2. *Let G be a locally finite 4-connected planar graph. Then G has a Hamilton circle.*

In this paper we will present a partial result in this direction:

Theorem 3. *Let G be a locally finite 6-connected planar graph that is k-indivisible for some finite $k \in \mathbb{N}$. Then G has a Hamilton circle.*

2 Definitions

In general, our notation will be that of Diestel [4]. If not otherwise noted, all graphs will be simple. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray or double ray are its *tails*. Let $G = (V, E)$ be any locally finite graph. Two rays in G are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G.

We say that a finite vertex set S *separates* an end ω from a vertex set U if every ray $R \in \omega$ that starts in a vertex of U meets S. In a similar manner, S separates two ends ω and ω' if every double ray with one tail in ω and the other in ω' goes through S.

We define a topology on G together with its ends, i.e. our topological space consists of all vertices, all inner points of edges and all ends of G. On G the topology will be that of a 1-complex. Thus, the basic open neighbourhoods of an inner point on an edge are the open intervals on the edge containing that point, while the basic open neighbourhoods of a vertex x are the unions of half-open intervals containing x, one from every edge at x. For every end ω and any finite set $S \subseteq V$ there is exactly one component $C = C(S, \omega)$ of $G - S$ which contains a tail of every ray in ω. We say that ω *belongs* to C, and write $\overline{C}(S, \omega)$

for the component C together with all the ends belonging to it. Then the basic open neighbourhoods of an end ω are all sets of the form

$$\hat{C}(S,\omega) := \overline{C}(S,\omega) \cup \mathring{E}(S,\omega)$$

$\mathring{E}(S,\omega)$ is the set of all inner points of edges between S and $C(S,\omega)$. This topological space will be denoted by $|G|$, and is also known as the *Freudenthal compactification* of G.

We will freely view G either as an abstract graph or as a subspace of $|G|$, i.e. the union of all vertices and edges of G with the usual topology of a 1-complex. It is not difficult to see that if G is connected and locally finite, then $|G|$ is compact. Note that in $|G|$ every ray converges to the end of which it is an element.

A set $C \subseteq |G|$ is a *circle* if it is homeomorphic to the unit circle. Then C includes every edge of which it contains an inner point, and the graph consisting of these edges and their endvertices is the *cycle* defined by C. Conversely, Diestel and Kühn [5] show that $C \cap G$ is dense in C, so every circle is the closure in $|G|$ of its cycle and hence defined uniquely by it. Note that every finite cycle in G is also a cycle in this sense, but there can also be infinite cycles; see Diestel [3, 4] for examples and for more information. A *Hamilton circle* is a circle that contains every vertex (and then also every end) of G.

G is called *planar* if there is an embedding of G (as a 1-complex) in the sphere S^2. Since we work in $|G|$ rather than in G, the following fact is quite convenient:

Theorem 4 (Richter and Thomassen [8]). *Let G be a locally finite 2-connected planar graph. Then $|G|$ embeds in the sphere.*

In an embedding $\varphi : |G| \to S^2$ in the sphere, every connected component of $S^2 \setminus \varphi(|G|)$ is called a *face*. Its boundary is a *face boundary*. It is not difficult to see that each face boundary is (or, more precisely, corresponds to) the closure of a subgraph of G.

Lemma 5 (Bruhn and Stein [2]). *Let G be a locally finite 2-connected graph with an embedding $\varphi : |G| \to S^2$. Then the face boundaries of $\varphi(|G|)$ are circles of $|G|$.*

For a subgraph H of G, an *H-bridge* is either a chord $e \notin E(H)$ together with its endvertices both of which lie in $V(H)$ or a component K of $G - H$ together with all edges between K and H, denoted by $E(K,H)$, and their incident vertices. We say that an H-bridge B is *chordal* if $E(B)$ consists of a single edge. All the vertices of an H-bridge B in H are *attachments* of B. For a subgraph F of G, we call a path (resp. cycle) P a *F-Tutte path* (resp. *cycle*) in G if every P-bridge of G has at most three attachments, and if every bridge containing an edge of F has at most two attachments.

3 Discussion

The Herschel graph (see Figure 2) is a well-known example for a 3-connected planar graph without a Hamilton cycle, which shows that in a sense Theorem 1 is best possible. In this section, let us briefly demonstrate that Conjecture 2 is also false for infinite graphs if we only assume 3-connectivity.

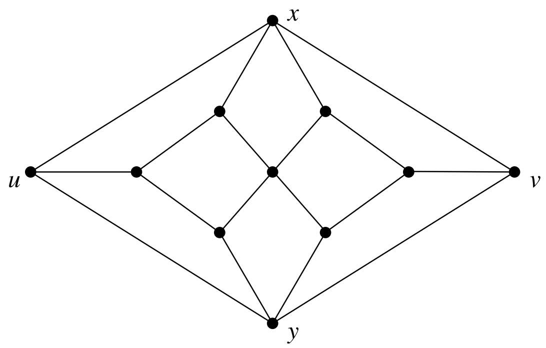

Figure 2: The Herschel graph

Consider Figure 3. There we have arranged copies of the Herschel graph (greyed) in a hexagonal grid. The copies are glued together in such a way that each copy of the vertex v in Figure 2 does not receive an extra edge and thus still has degree 3. Now assume that the resulting graph has a Hamilton circle C, and consider a copy of the Herschel graph H in that graph. If the Hamilton circle enters H in u and leaves H in either x or y then C induces a Hamilton circle of H, which is impossible. Thus, C enters H in x and leaves it in y, which implies that H has a Hamilton path between x and y. However, this is impossible since H is an odd bipartite graph, where x and y are in the same partition class.

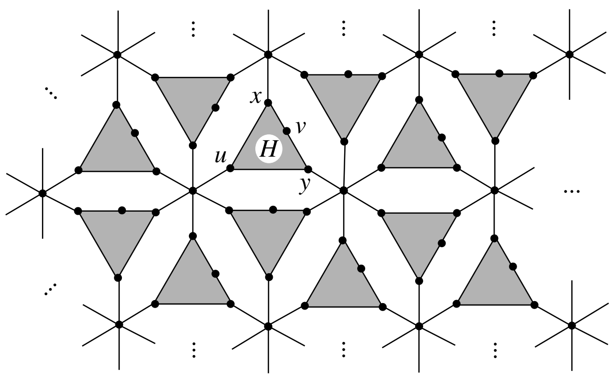

Figure 3: An infinite 3-connected planar graph without a Hamilton circle

4 Proof of main result

Before we start proving Theorem 3, let us reformulate the theorem slightly. The notion of ends are central to the definition of infinite cycles, and we will therefore express the theorem in terms of ends. It is straightforward to see that a locally finite graph is k-indivisible if and only if it has at most $k-1$ ends. Thus, we obtain the following alternative version of Theorem 3:

Theorem 6. *Let G be a locally finite 6-connected planar graph with at most finitely many ends. Then G has a Hamilton circle.*

Our main tool in the proof will be the following result of Thomassen, which itself implies Tutte's theorem:

Theorem 7 (Thomassen [9]). *Let G be a finite 2-connected plane graph with a face boundary C. Assume that $u \in V(C)$, $e \in E(C)$ and $v \in V(G) \setminus \{u\}$. Then G contains a C-Tutte path P from u to v and through e.*

In order to make use of Thomassen's theorem, we need to chop off of the graph all infinite parts, so that a finite part remains, in which we may apply Theorem 7. We then extend this finite Hamilton cycle to a finite part of the chopped off infinite components. For this to work, we need that we can separate off 3-connected infinite components from some arbitrary finite vertex set. This is the task of our next lemma, and, in particular, of its consequence, Lemma 10.

Lemma 8. *Let G be a 4-connected locally finite planar graph with minimum degree at least 6, let $U \subseteq V(G)$ be a finite vertex set, and let ω be an end of G. Then, there is a finite vertex set S such that for $C_\omega := C(S, \omega)$ holds*

(i) *C_ω is disjoint from U; and*

(ii) *if $X \subseteq V(C_\omega)$ with $|X| \leq 2$, then every component of $C_\omega - X$ is infinite.*

Proof. Choose a finite vertex set S such that $C_\omega := C(S, \omega)$ is disjoint from U, and such that $|E(C_\omega, G - C_\omega)|$ is minimal with that property.

Suppose there is a set $X \subseteq V(C_\omega)$ with $|X| \leq 2$ such that there is a finite component K of $C_\omega - X$. Denote by K' the subgraph obtained by adding the vertices in X and the edges between X and K to K. Put $r := |E(K, G - C_\omega)|$, and $s := |E(X, K)|$. We will show that $s < r$. Then $S' := S \cup V(K)$ leads to a smaller cut between $C(S', \omega)$ and the rest of the graph, a contradiction.

First, assume that K' is not a triangulation. Thus, putting $n := |V(K)|$, Euler's formula implies $|E(K')| < 3(n + |X|) - 6 \leq 3(n + 2) - 6 = 3n$. On the other hand,

$$2|E(K')| = \sum_{v \in V(K')} d_{K'}(v) = |E(X, K)| + \sum_{v \in V(K)} d_K(v) \geq s + 6n - r.$$

For the last inequality recall that the minimum degree of G is at least 6. Hence, $3n + (s - r)/2 \leq |E(K')| < 3n$ and thus $s < r$, as desired.

Second, let K' be a triangulation. We may assume that some vertices in S lie in the outer face of K'. Since G is 4-connected and since the face boundary of the outer face of K' is a triangle, T, say, no vertices of G can be contained in the interior face of T. Thus, $T = K'$, and consequently the set X consist of exactly one vertex as there are no edges between vertices of X in K'. This implies that there are exactly two edges between X and K, ie. $s = 2$. Since the minimum degree is at least 6 in G, and as the exactly two vertices in K can have at most one edge between them, it follows that $r \geq 3$. Again, we get $s < r$, as desired. □

Lemma 9. *Let G be a k-connected graph, and let $V(G) = A \cup B$ be a partition such that $G[B]$ is l-connected. Consider $X \subseteq A$ with $|X| \leq k - l$. Then, for every component K of $G[A] - X$ the graph $G[K \cup B]$ is still l-connected.*

Proof. Put $B' := V(K) \cup B$, and suppose there is a separator Y of $G[B']$ with $|Y| < l$, and consider two distinct components C and D of $G[B'] - Y$. If both C and D contain vertices of B then $Y \cap B$ is a separator of $G[B]$, contradicting that $G[B]$ is l-connected. So we may assume that $C \subseteq K$. Then, $X \cup Y$ separates C from B in G, but $|X \cup Y| < (k-l) + l = k$, a contradiction. $\square$

Lemma 10 will be used in each of the induction steps of the proof of Theorem 6.

Lemma 10. *Let G be a locally finite planar 6-connected graph, let U be a finite vertex set and ω an end of G. Let $S \subseteq V(G)$ be a finite vertex set such that $C := C(S, \omega)$ is 2-connected. Then there is a finite vertex set $T \supseteq U$ such that $C(T, \omega) \subseteq C$ is 3-connected and $C - C(T, \omega)$ 2-connected.*

Proof. Add to the union of $U \cap V(C)$ and the neighbours of S in C finitely many vertices such that for the resulting set $U' \subseteq V(C)$ the subgraph $G[U']$ is 2-connected. Then Lemma 8 applied to $U' \cup S$ yields a finite vertex set T' so that $D := C(T', \omega) \subseteq C$ is 3-connected and disjoint from U' (and thus from U). Since $G[U'] \subseteq C$ is 2-connected there is a block B of $C - D$ containing U'. Observe that, as G is 6-connected, every component of $C - D - B$ has a neighbour in the finite set $S \cup T'$. Thus there are only finitely many of them. Consider one such component, K say, and note that, as K is disjoint from U' all its neighbours except possibly one, which lies in B, are contained in D. Thus, by Lemma 9, $G[D \cup K]$ is still 3-connected. Therefore, the neighbours of $C - B$ together with U give rise to a set T as desired. $\square$

We will construct our Hamilton circle in a piecewise manner. Slightly more precise, we will construct finite nested subgraphs G_i in which the application of Theorem 7 will yield subgraphs H_i that are finite approximations of the desired infinite Hamilton circle. The following definition and lemma make sure that these subgraphs H_i indeed tend to a cycle, ie. that $\bigcup_{i=1}^{\infty} H_i$ is a cycle.

Call a sequence $(G_1, H_1), \ldots, (G_k, H_k)$ of finite induced subgraphs $G_i \subseteq G$ and subgraphs $H_i \subseteq G$ *good*, if for $i = 1, \ldots, k$ holds

(i) $H_{i-1} \subseteq H_i$ and $G_{i-1} \cup N(G_{i-1}) \subseteq G_i$ for $i \geq 2$;

(ii) if K is an infinite component of $G - G_i$ then $|E(K, G - K) \cap E(H_i)| = 2$; and

(iii) there is a cycle Z such that $Z \cap G_i = H_i \cap G_i$.

Lemma 11. *Let G be a locally finite graph with at most finitely many ends, and let every finite initial segment of $(G_1, H_1), (G_2, H_2), \ldots$ be good. Then the closure of $H := \bigcup_{i=1}^{\infty} H_i$ is a circle of $|G|$.*

Proof. From (iii) and (i) it follow that H is a either a finite cycle or a disjoint union of double rays. Assume the latter, and let ω be an end of one of the double rays. Because of (i) there is an n such that the component K of $G - G_n$ to which ω belongs is separated from all the other ends by G_n. Condition (ii) implies that $H \cap K$ consists of exactly two rays in ω, which shows that $\overline{H}$ is the disjoint union of circles. It is easy to see that (iii) implies that $\overline{H}$ is topologically connected. Therefore, $\overline{H}$ is a circle. $\square$

Let us introduce one useful definition before we finally start with the proof. We call a vertex set U in a graph G *externally k-connected*, if $|U| \geq k$ and if for every set $X \subseteq V(G)$ with $|X| < k$ there is a path between any two vertices of $U \setminus X$ in $G - X$.

Proof of Theorem 6. By Theorem 4, $|G|$ can be embedded in the sphere. We will identify $|G|$ with that embedding, and thus view G as a plane graph.

Inductively, we will construct connected finite induced subgraphs $G_1 \subseteq G_2 \subseteq \ldots \subseteq G$ and subgraphs $H_1 \subseteq H_2 \subseteq \ldots \subseteq G$ such that for every $i \geq 1$ it holds that

(iv) $(G_1, H_1), \ldots, (G_i, H_i)$ is good;

(v) $V(G_i) \subseteq V(H_i)$;

(vi) if $\mathcal{C}_i$ is the set of components of $G - G_i$ then each $C \in \mathcal{C}_i$ is 3-connected and belongs to exactly one end; and

(vii) for each $C \in \mathcal{C}_i$ it follows that $|V(C) \cap V(H_i)| = 2$.

Assume this can be achieved. Then, Lemma 11 shows that $H := \overline{\bigcup_{i=1}^{\infty} H_i}$ is a circle. In addition, H contains every vertex of G, by (v) and (i). Therefore, H is a Hamilton circle.

As the construction of the base case, i.e. when $i = 1$, is quite similar to the general case, i.e. when $i \geq 2$, we will treat both at once. However, for some of the steps we will need to make case distinctions. Put $G_0 := \emptyset$, $H_0 := \emptyset$ and $\mathcal{C}_0 = \{G\}$, and assume (G_{i-1}, H_{i-1}) to be constructed for some $i \geq 1$. Consider a component $C \in \mathcal{C}_{i-1}$.

First, let $i = 1$, and note that as $C = G$ has only finitely many ends, we can choose a finite vertex set W such that none of the infinite components of $G - W$ belong to more than one end. Denote by F_C a face boundary of $|G|$, and pick an edge $x_C y_C \in E(F_C)$, a vertex $u_C \in V(F_C)$ with $u_C \notin \{x_C, y_C\}$, and a vertex $v_C \notin \{u_C, x_C, y_C\}$ that is adjacent to u_C. Put $U_C := W \cup \{u_C, v_C, x_C, y_C\}$.

For $i \geq 2$, observe that, by (vii), H_{i-1} contains exactly two vertices of C, u_C and v_C say. In the embedding of $\overline{C}$ induced by $|G|$ all the neighbours of the connected graph G_{i-1} that lie in C are incident with the same face boundary F_C, say. Pick an edge $x_C y_C \in E(F_C)$ such that $x_C y_C \neq u_C v_C$, and let U_C be the union of $\{u_C, v_C, x_C, y_C\}$ together with all neighbours of G_{i-1} in C. Thus, $U_C \subseteq V(F_C)$ separates C from the rest of G.

In any case, we obtain

$$u_C \neq v_C \textit{ and } |\{u_C, v_C, x_C, y_C\}| \geq 3 \textit{ (resp. } = 4 \textit{ for } i = 1\textit{).} \tag{1}$$

Since C is 3-connected (resp. 4-connected for $i = 1$) there are three (resp. four) internally disjoint paths in C between any two vertices in U_C. Denote by T_C the vertex set of the union of three (resp. four) such paths for each pair of vertices in U_C. Observe that U_C is externally 3-connected (resp. externally 4-connected for $i = 1$) in $C[T_C]$.

Let $\omega_1, \ldots, \omega_n$ be an enumeration of the ends belonging to C. Put $C_1 := C$, and apply Lemma 10 to C_1, ω_1, T_C in order to find a finite vertex set S_1 for which $C(S_1, \omega_1) \subseteq C_1$ is 3-connected and disjoint from T_C and for which $C_2 := C_1 - C(S_1, \omega_1)$ is 2-connected. Continuing in this manner, we see that

$$G'_C := C - \bigcup_{i=1}^{n} C(S_i, \omega_i) \text{ is a finite 2-connected graph, and } U_C \text{ is externally 3-connected (resp. 4-connected for } i = 1\text{) in } G'_C. \tag{2}$$

Since $U_C \subseteq V(G'_C)$ contains all neighbours of G_{i-1} in C it follows that

$$G'_i := G[V(G_{i-1} \cup \bigcup_{B \in \mathcal{C}_{i-1}} G'_B)] \text{ is connected, and every component of } G - G'_i \text{ is 3-connected and belongs to exactly one end of } G. \tag{3}$$

G'_i will serve as a precursor to G_i.

Next, let $\mathcal{D}_C$ be the components D of $G - G'_i$ with $D \subseteq C$, and consider $D \in \mathcal{D}_C$. (Note that for $i > 1$ it holds that $\mathcal{D} = \{D\}$). Let M_D be obtained from $E(D, G - D)$ by deleting for each vertex in D all but one of the incident edges in $E(D, G - D)$. Thus,

$$\text{every neighbour of } G'_C \text{ in } D \text{ is incident with exactly one edge of } M_D. \tag{4}$$

Starting from G'_C define a 2-connected finite plane graph $\tilde{G}_C$ as follows: for every $D \in \mathcal{D}_C$ put a vertex z_D into the face of G'_i that contains D and link z_D to the vertices in G'_C that are incident with an edge in M_D. We will identify these linking edges with the edges in M_D. Note that the resulting graph, which is a minor of G, may have parallel edges. Then, since, for $i = 1$, F_C is a face boundary of $|G|$ such that $u_C, x_C, y_C \in V(F_C)$ and since, for $i \geq 2$, $U_C \subseteq V(F_C)$ this also holds for a face boundary $\tilde{F}_C$ of $\tilde{G}_C$, i.e.

$$\begin{array}{l} \text{for } i = 1: \ u_C, x_C, y_C \in V(\tilde{F}_C) \text{ and } U_C \text{ separates the ends of } G \\ \text{for } i \geq 2: \ U_C \subseteq V(\tilde{F}_C) \text{ and } U_C \text{ separates } C \text{ from } G - C \end{array} \tag{5}$$

We apply Theorem 7 to $\tilde{G}_C$, and obtain a $\tilde{F}_C$-Tutte path (resp. circle for $i = 1$) $\tilde{H}_C$ from u_C to v_C (resp. through $u_C v_C$) and through $x_C y_C$. (More precisely, if $\tilde{G}_C$ has parallel edges we first subdivide these before using Theorem 7; the obtained Tutte-path then induces one in $\tilde{G}_C$.) From (1) it follows that

$$|V(\tilde{H}_C) \cap U_C| \geq 3 \text{ (resp. } \geq 4 \text{ for } i = 1\text{)}. \tag{6}$$

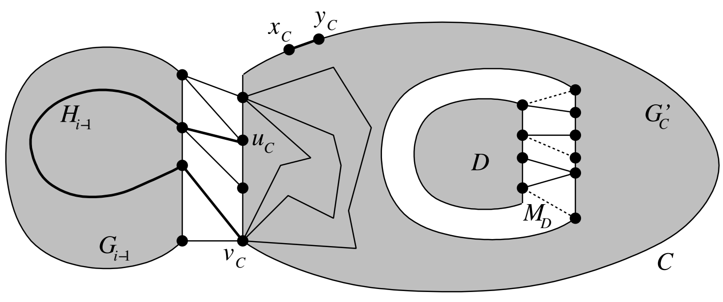

Figure 4: Illustration of the proof of Theorem 6

Next, we show that

$$\text{for every nonchordal } \tilde{H}_C\text{-bridge } K \text{ in } \tilde{G}_C, (K - \tilde{H}_C) \cap G \text{ is disjoint from } U_C, \text{ and all its neighbours in } G \text{ lie in } G'_C \cup D \text{ for some } D \in \mathcal{D}_C. \tag{7}$$

Suppose that $K-\tilde{H}_C$ meets U_C. If $i \geq 2$ then K contains an edge of $\tilde{F}_C$, as $U_C \subseteq V(\tilde{F}_C)$, by (5). Thus, K has at most two (resp. three for $i=1$) attachments as $\tilde{H}_C$ is a $\tilde{F}_C$-Tutte path (resp. cycle). Since U_C is externally 3-connected (resp. externally 4-connected) in G'_C by (2), this implies $U_C \subseteq V(K)$. Thus, by (6), $K-\tilde{H}_C$ contains a vertex in $V(\tilde{H}_C) \cap U_C$, a contradiction. Therefore, $K-\tilde{H}_C$ is disjoint from U_C. The second assertion follows from the first together with (5) and the fact that $|\mathcal{D}_C| = 1$ if $i \geq 2$.

It holds that

$$z_D \in V(\tilde{H}_C) \text{ for every } D \in \mathcal{D}_C. \tag{8}$$

Indeed, suppose there is a $D \in \mathcal{D}_C$ with $z_D \notin V(\tilde{H}_C)$. Thus, $z_D \in V(K-\tilde{H}_C)$ for some $\tilde{H}_C$-bridge K. Denote by f_D the face of G'_i that contains D. Then there exists a vertex $v \notin V(K)$ in the face boundary of f_D. Indeed, otherwise the at most three attachments of K separate D from $U_C \setminus V(K) \neq \emptyset$ in G, contradicting that G is 6-connected. Note that v is not incident with any edge in M_D as $z_D \in V(K-\tilde{H}_C)$. Let $f \subseteq f_D$ be the (unique) face of $G'_i \cup D \cup M_D$ whose face boundary F contains v. Because G'_i and D are connected there are exactly two vertices in $V(F \cap D)$, a and b say, that are incident with an edge in M_D.

We claim that the at most three attachments of K together with a, b separate $D-\{a,b\}$ from v in G, which then contradicts that G is 6-connected, thus establishing (8). So suppose there is a path from D to v that avoids the attachments of K and a, b, and let xy be its last edge in $E(D, G-D) = E(D, G'_i)$. Assume that $x \in V(D)$ and $y \in V(G'_i)$. If xy is disjoint from f then y is easily seen to be separated in G'_i from v by the attachments of K, which is impossible. Therefore, the interior of xy lies in f, and thus x in its boundary F. By (4), x is incident with an edge in M_D. Since a, b are the only vertices in $V(F \cap D)$ incident with an edge in M_D, we obtain with $x \in \{a,b\}$ a contradiction. This establishes (8).

Consider $E(\tilde{H}_C)$ as a subset of $E(G)$, and let H_C be the subgraph of G consisting of the edges $E(\tilde{H}_C)$ and the incident vertices. Put $H_i := H_{i-1} \cup \bigcup_{C \in \mathcal{C}_{i-1}} H_C$, and observe that the pair (G'_i, H_i) already satisfies almost all of the desired properties. In particular, it satisfies (i)–(iii) of the definion of a good sequence: (i) holds because of $U_C \subseteq V(G'_C)$ for every $C \in \mathcal{C}_{i-1}$ (for $i \geq 2$), and (ii) because of (8). To see (iii), add a path through D between the two vertices of H_i in D for every $D \in \mathcal{D}_C$, and denote the resulting subgraph by Z. Clearly, Z is connected, and every $w \in V(Z)$ has degree two: if $w \in V(G_{i-1})$ then because of (iii) for $i-1$, if $w \in \{u_C, v_C\}$ then because $u_C \neq v_C$, if $w \in V(H_i - H_{i-1})$ because H_i is a disjoint union of paths for every C, and finally if $w \in V(Z - H_i)$ because $Z - H_i$ is a disjoint union of paths. Thus, Z is a cycle. Moreover, of (iv)–(vii) only (v) is not satisfied: (vi) is (3), and (vii) holds because of (8) and the definition of M_D.

To fix (v), consider a nonchordal H_i-bridge K in G'_i. By (v) for $i-1$, we deduce that $K - H_i$ is disjoint from G_{i-1}. Observe that for each $C \in \mathcal{C}_{i-1}$, $K \cap G'_C$ is either empty, a chord or a union of H_C-bridges. Thus, $K - H_i$ is disjoint from U_C, by (7), which implies with Lemma 9 that $G'_i - (K - H_i)$ still satisfies (i) and (vi), and then also (iv) and (vii). Thus, putting $G_i := G'_i - (G'_i - H_i)$ we see that for the pair (G_i, H_i) conditions (iv)–(vii) hold. $\square$

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Henning Bruhn `<hbruhn@gmx.net>`
Mathematisches Seminar
Universität Hamburg
Bundesstraße 55
20146 Hamburg
Germany

Xingxing Yu`<yu@math.gatech.edu>`
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
USA

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