# Forcing highly connected subgraphs

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**Abstract**

By a theorem of Mader [5], highly connected subgraphs can be forced in finite graphs by assuming a high minimum degree. Solving a problem of Diestel [2], we extend this result to infinite graphs. Here, it is necessary to require not only high degree for the vertices but also high *vertex-degree* (or *multiplicity*) for the ends of the graph, i.e. a large number of disjoint rays in each end.

We give a lower bound on the degree of vertices and the vertex-degree of the ends which is quadratic in $k$, the connectedness of the desired subgraph. In fact, this is not far from best possible: we exhibit a family of graphs with a degree of order $2^k$ at the vertices and a vertex-degree of order $k \log k$ at the ends which have no $k$-connected subgraphs.

Furthermore, if in addition to the high degrees at the vertices we only require high *edge-degree* for the ends (which is defined as the maximum number of edge-disjoint rays in an end), Mader's theorem does not extend to infinite graphs, not even to locally finite ones. We give a counterexample in this respect. But, assuming a lower bound of at least $2k$ for the edge-degree at the ends and the degree at the vertices does suffice to ensure the existence $(k+1)$-*edge*-connected subgraphs in arbitrary graphs.

# 1 Introduction

In a finite graph, high average degree forces the existence of a highly connected subgraph:

**Theorem 1.1** (Mader [5])**.** *Any finite graph $G$ of average degree at least $4k$ has a $(k+1)$-connected subgraph.*

In infinite graphs, it is not clear what an adequate concept of 'average degree' should be. In this paper, we restrict ourselves to investigate the consequences of the (in finite graphs stronger) assumption of 'high minimum degree'.

But simply requiring high degree for the vertices is not enough, as the counterexample of the infinite $r$-regular tree $T^r$ demonstrates. Now, since an infinite tree has rather 'thin' ends (which seem to play the role of the leaves of the infinite tree), this suggests, as conjectured by Diestel [2], that a minimum degree condition has to be imposed also on the ends of the graph.

In fact, let us define the *vertex-degree* of an end as the maximum number of disjoint rays in it (this maximum exists, see Section 2). Then $T^r$ ceases to be a counterexample, as each of its ends has vertex-degree 1. And indeed, with this further condition on the vertex-degrees of the ends, highly connected subgraphs can be forced also in infinite graphs. As our main theorem we prove the following stronger result (for this, let us call an induced connected subgraph of a graph

$G$ with finite boundary a *region*; note that in particular, any component of $G$ is a region):

**Theorem 5.1.** *Let $k \in \mathbb{N}$ and let $G$ be a graph such that each vertex has degree at least $2k(k+3)$, and each end has vertex-degree at least $2k(k+1)+1$. Then every infinite region of $G$ has a $(k+1)$-connected region.*

Observe that while in Theorem 1.1, the bound on the degrees is linear in $k$, in Theorem 5.1 we require quadratic degree in $k$. This is in fact near to best possible:

**Theorem 6.1.** *For each $k = 5\ell$, where $\ell \in \mathbb{N}$, there exists a locally finite graph whose vertices have degree at least $2^{\ell}$ and whose ends have vertex-degree at least $\ell \log \ell$, and which has no $(k+1)$-connected subgraph.*

A similar way to extend the degree notion to ends is the *edge-degree*, which, as suggested in [1], is defined as the maximum number of edge-disjoint rays in the end (the maximum exists, see Section 2). It seems that these two concepts reflect different aspects of the degree of a vertex. The vertex-degree of an end is the analogue of the size of the neighbourhood of a vertex, while the edge-degree corresponds to the number of incident edges.

This point of view suggests that for forcing highly (*vertex-*)connected subgraphs, high vertex-degree is a more natural requirement than high edge-degree. And in fact, it turns out that high edge-degrees at the ends and high degrees at the vertices together are not sufficient to force highly connected subgraphs, or even highly connected minors, in infinite graphs. In Section 4 we exhibit for all $r \in \mathbb{N}$ a locally finite graph of minimum degree and minimum edge-degree $r$ that has no 4-connected subgraph and no 6-connected minor.

But, the assumption of high degree and high edge-degree does suffice to force highly edge-connected subgraphs in arbitrary graphs, with a lower bound on the (edge-)degrees that is only linear in $k$:

**Theorem 3.1.** *Let $k \in \mathbb{N}$ and let $G$ be a graph such that each vertex has degree at least $2k$ and each end has edge-degree at least $2k$. Then $G$ has a $(k+1)$-edge-connected region.*

Moreover, highly edge-connected subgraphs can be found in every infinite region (Theorem 3.3).

In general, it is not possible to force *finite* highly vertex-/edge-connected subgraphs in infinite graphs by assuming high minimum degree and vertex- resp. edge-degree. Neither can one force *infinite* highly vertex- or edge-connected subgraphs. Counterexamples in this respect are provided in the discussion after Corollary 3.4, near the end of Section 3. However, any graph which obeys the (vertex-/edge-)degree bounds of Theorem 5.1 resp. Theorem 3.1, has either an infinite $(k+1)$-vertex-/edge-connected subgraph or infinitely many finite such (see Corollary 3.4/Corollary 5.3).

A related question is whether large complete minors can be forced by assuming a high minimum degree and vertex-degree.[1] Note that again some condition on the ends is necessary, because of the trees $T^r$ for large $r$. However, high

[1] There are other traditional ways to force large complete minors in finite graphs. One of these is the assumption of large girth $g$ plus a minimal degree of at least 3. This does not

minimal degree and vertex-degree is not even enough to force a $K^4$, which we can see by adding a spanning path to each level of $T^r$: the resulting planar graph has a single end of infinite vertex-degree. This observation was already made in [1].

This paper is organised as follows. After giving some elemetary definitions in Section 2, we prove the edge-version of Mader's theorem for arbitrary graphs, Theorem 3.1, and some related results in Section 3. In Section 4, we show that high degree at the vertices and high edge-degree at the ends does not force highly (*vertex-*)connected subgraphs. Section 5 is devoted to the proof of our main theorem, Theorem 5.1, whose quadratic bounds on the (vertex-)degree we justify in Section 6, where we exhibit a family of graphs with degrees of order $2^k$ and vertex-degrees of order $k \log k$, which have no $(k+1)$-connected subgraphs.

# 2 Terminology

The terminology we use is standard, and can be found for example in [3]. A 1-way infinite path is called a *ray*, and the subrays of a ray are its *tails*. Two rays in a graph $G$ are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of $G$. We denote the set of the ends of $G$ by $\Omega(G)$.

Let $H$ be a (possibly empty) subgraph of $G$, and write $H \subseteq G$. The *boundary* $\partial H$ of $H$ is the set $N(G-H)$ of all neighbours in $H$ of vertices of $G-H$. We call $H$ a *region* (of $G$) if $H$ is a connected induced subgraph with finite boundary. Then $H' \subseteq H$ is a region of $G$ if and only if it is a region of $H$.

Call a region $H$ *profound*, if $V(H) \neq \partial H$. For example, all infinite regions are profound, and a profound region is not empty.

As in finite graphs, we call $H$ *$k$-connected* for some $k \in \mathbb{N}$, if $|H| > k$ and no set of fewer than $k$ vertices separates $H$. Similarly, $H$ is *$k$-edge-connected* if $|H| > 1$ and no set of fewer than $k$ edges separates $H$. Hence, if $H$ is not $k$-edge-connected (and non-trivial), then it has a cut of cardinality less than $k$.

We shall consider two different extensions of the degree notion to ends. The *vertex-degree* (also known as the *multiplicity*, or *thickness*) of an end $\omega \in \Omega(G)$ is the maximum cardinality of a set of (vertex-)disjoint rays in $\omega$. The *edge-degree* of $\omega$ (as suggested in [1]) is the maximum cardinality of a set of edge-disjoint rays in $\omega$. It can be shown that these two degree concepts are well-defined, i.e. the considered maxima do indeed exist.[2]

extend to infinite graphs: Take the union over all $n \in \mathbb{N}$ of the cycles $C_{g^n}$, and add edges between each $C_{g^n}$ and $C_{g^{n+1}}$ so that each vertex of $V(C_{g^n})$ has a neighbour in $V(C_{g^{n+1}})$, and these neighbours have distance $g$ from each other. This yields a locally finite planar graph of girth $\geq g$ and minimum degree and vertex-degree $\geq 3$.

Another option for ensuring large complete minors in finite graphs is a large chromatic number, but this trivially carries over to infinite graphs. Any infinite graph $G$ has a finite subgraph $H$ with the same chromatic number (this is a well-known theorem of de Bruijn and Erdős, see e.g. [3] for a proof and references). Then the desired large complete minor is already contained in $H$.

[2]Halin [4] proves the existence of an infinite set of disjoint rays if the number of disjoint rays in the considered graph is unbounded: with slight modifications, his proof yields the same result for rays of a fixed end. In [1], it is shown that the supremum of the cardinalities of sets of edge-disjoint rays in a given end is attained in locally finite graphs: this proof carries over similarly to arbitrary graphs.

## 3 Forcing highly edge-connected subgraphs

We start by proving our second result, Theorem 3.1, which is easier:

**Theorem 3.1.** *Let $k \in \mathbb{N}$ and let $G$ be a graph such that each vertex has degree at least $2k$ and each end has edge-degree at least $2k$. Then $G$ has a $(k+1)$-edge-connected region.*

Theorem 3.1 is best possible in the sense that high edge-degree is not sufficient to force highly connected subgraphs, as we shall see in the next section.

For the proof, we need the following lemma, which basically assures that if a graph contains some region with small cut to the outside world, then there is either a 'smallest' such, or we have an infinite nested sequence of such regions so that their cuts are all disjoint.

**Lemma 3.2.** *Let $D \neq \emptyset$ be a region of a graph $G$ so that $|E(D, G-D)| < m$ and so that $|E(D', G-D')| \geq m$ for every non-empty region $D' \subseteq D - \partial D$ of $G$. Then there is an inclusion-minimal region $H \subseteq D$ with $|E(H, G-H)| < m$ and $H \neq \emptyset$.*

*Proof.* If there is no such $H$, then we can construct an infinite sequence of distinct regions $D =: D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \ldots$ such that all cuts $F_i := E(D_i, G-D_i)$ have cardinality less than $m$. Note that any edge that lies in some $F_i$, but not in $F_{i+1}$, lies outside $E(D_{i+1}) \cup F_{i+1}$, and hence will not appear in any $F_j$ with $j > i$.

By assumption, every region $D' \subseteq D$ which is not incident with any edge of $F_0$, sends at least $m$ edges to the outside. Thus, there is an edge $e$ in $F_0$ that appears in all $F_i$ for $i \geq 0$. Let $E$ be the set of all edges $e_j$ for which there exists an index $j$ such that $e_j \in F_i$ for all $i \geq j$. Clearly, $e \in E$, and $|E| < m$, where the latter follows from the boundedness of the cuts $F_i$.

Let $n$ be so that $E \subseteq F_n$. Now, as $D_{n+1} \subsetneq D_n$, there is a vertex $x \in V(D_n - D_{n+1})$. Since $D_n$ is connected, it contains a (finite) path $P$ that connects $x$ with $y$, the endvertex of $e$ in $D_n$. All $D_i$ with $i > n$ contain $y$, but not $x$, thus each $F_i$ with $i > n$ must contain one of the edges on $P$. This implies that there is an edge $e_j$ on $P$ which for some $j > n$ lies in all $F_i$ with $i \geq j$. Thus, $e_j \in E \subseteq F_n$, but $F_n \cap E(P) = \emptyset$, a contradiction. □

We now prove Theorem 3.1.

*Proof of Theorem 3.1.* First of all, we shall show that there exists a region $C \neq \emptyset$ such that

*(a)* $|E(C, G-C)| < 2k$, *and*

*(b) for every non-empty region* $C' \subseteq C - \partial C$ *holds* $|E(C', G-C')| \geq 2k$.

Indeed, let us construct a sequence $C_0 \supseteq C_1 \supseteq C_2 \ldots$ of non-empty regions such that for $i \geq 0$ the following hold

(i) $|E(C_i, G-C_i)| < 2k$, and

(ii) $C_{i+1} \subseteq C_i - \partial C_i$.

Choose $C_0$ as any component of $G$. Now, if after finitely many, say $j$, steps of our construction we cannot go on, i.e. find a suitable $C_{j+1}$, it is because $C_j$ has property (b). Property (a) is then ensured by (i).

So assume that we end up with an infinite sequence $C_0, C_1, C_2, \ldots$ of regions. Observe that, since $C_0$ is a region, $\partial C_i \neq \emptyset$ for each $i \geq 1$. As each of the $C_i$ is connected, there is a sequence $(P_i)_{i\in\mathbb{N}}$ of $\partial C_i$–$\partial C_{i+1}$ paths such that for $i \geq 1$ the path $P_{i+1}$ starts in the last vertex of $P_i$. By (ii), the paths $P_i$ are non-trivial, and by construction, each $P_i$ meets $P_{i-1}$ and $P_{i+1}$ only in its first resp. last vertex, and is disjoint from all the other $P_j$. Hence, their union $P := \bigcup_{i=1}^{\infty} P_i$ is a ray which has a tail in each of the $C_i$.

Let $\omega$ be the end of $G$ that contains $P$. As, by assumption, $\omega$ has edge-degree at least $2k$, there is a family $\mathcal{R}$ of $2k$ edge-disjoint $\omega$-rays in $G$. For each ray $R \in \mathcal{R}$ let $n_R$ denote the distance its starting vertex has to $\partial C_1$. Set $n := \max\{n_R : R \in \mathcal{R}\} + 2$. Then by (ii), all of the $2k$ disjoint rays in $\mathcal{R}$ start outside $C_n$. But each ray in $\mathcal{R}$ is equivalent to $P$, and hence eventually enters $C_n$, contradicting the fact that $|E(C_n, G - C_n)| < 2k$ by (i). This proves the existence of a region $C \neq \emptyset$ with the properties (a) and (b).

Thus, Lemma 3.2 yields an inclusion-minimal non-empty region $H \subseteq C$ with $|E(H, G - H)| < 2k$. We claim that $H$ is the desired $(k + 1)$-edge-connected region of $G$. In fact, otherwise the bound on the degrees of the vertices of $G$ implies that $|H| \geq 2$, and so, $H$ has a cut $F$ with $|F| \leq k$. We may assume that $F$ is a minimal cut, i.e. splits $H$ into two (non-empty) regions $H'$ and $H''$. For one of the two, say $H'$, the cut $E(H', G - H')$ meets $E(H, G - H)$ in at most $\frac{|E(H,G-H)|}{2} < k$ edges. Hence

$$|E(H', G - H')| \leq |E(H', G - H') \cap E(H, G - H)| + |F| < 2k$$

and $H' \subsetneq H$, contradicting the minimality of $H$. □

Note that our proof yields a $(k+1)$-edge-connected region in every region $C$ of $G$ with $|E(C, G - C)| < 2k$ (simply start with $C_0 := C$ instead of taking any component of $G$). With slightly more effort (and slightly higher edge-degree), one can prove that every *infinite* region of $G$ contains a $(k + 1)$-edge-connected region:

**Theorem 3.3.** *Let $k \in \mathbb{N}$, and let $G$ be a graph such that each vertex has degree at least $2k$, and each end has edge-degree at least $2k + 1$. Then every infinite region of $G$ contains a $(k + 1)$-edge-connected region.*

*Proof.* Let $D$ be an infinite region of $G$. If there is a region $D' \subseteq D$ with $|E(D', G - D')| \leq 2k$ and $D \neq \emptyset$, then we proceed as in the proof of Theorem 3.1 to find an inclusion-minimal non-empty region $H$ with this property, which then turns out to be the desired $(k + 1)$-edge-connected region.[3]

So, we can assume that $D$ contains no non-empty region which sends less than $2k + 1$ edges to the outside. Now, let $H \subseteq D$ be an infinite region with $|E(H, G - H)|$ minimal. If we can prove $H$ to be $(k+1)$-edge-connected, we are done. But otherwise there is a cut $F$ with $|F| \leq k$ that splits $H$ into two regions $H'$ and $H''$. At least one of these, say $H'$, is infinite. By the choice of $H$, the

[3]Observe that, starting with $|E(D', G - D')| \leq 2k$ instead of $< 2k$, we will have to adjust our inequalities, and the final contradiction is obtained by finding that $|E(H', G - H')| \leq 2k$.

number of edges $H'$ sends to the rest of the graph is at least $|E(H, G-H)|$; hence, $E(H', G-H')$ contains all but at most $|F|$ edges of $E(H, G-H)$. Thus,

$$|E(H'', G-H'')| = |F \cup (E(H, G-H) - E(H', G-H'))| \leq 2k,$$

contradicting our assumption on $D$. □

Theorem 3.3 has two interesting corollaries.

**Corollary 3.4.** *Let $k \in \mathbb{N}$ and let $G$ be a graph in which all vertices have degree at least $2k$, and all ends have edge-degree at least $2k+1$. Then any infinite region $C$ of $G$ has either infinitely many disjoint finite $(k+1)$-edge-connected regions, or an infinite $(k+1)$-edge-connected region.*

*Proof.* Take an inclusion-maximal set $\mathcal{D}$ of disjoint finite $(k+1)$-edge-connected regions of $C$ (which exists by Zorn's Lemma), and assume that $|\mathcal{D}| < \infty$. Since any infinite component of $C' := C - \bigcup_{D \in \mathcal{D}} D \subseteq C$ is an infinite region of $G$, we may use Theorem 3.3 to obtain a $(k+1)$-edge-connected region $H$ of $C$. Then $H$ is infinite by the choice of $\mathcal{D}$. □

The two configurations of Corollary 3.4 of which one necessarily appears in any given graph of large enough minimal (edge-)degree, need not both exist, not even in locally finite graphs. Indeed, for given $r \in \mathbb{N}$, it is easy to construct an infinite locally finite graph $G$ which has minimum degree and vertex- (and thus edge-) degree $r$ but no infinite 3-edge-connected subgraph. We obtain $G$ from the $r \times \mathbb{N}$ grid by joining each vertex to $r$ disjoint copies of $K^{r+1}$. Any infinite subgraph of $G$ which is at least 2-edge-connected is also a subgraph of the $r \times \mathbb{N}$ grid, and hence is at most 2-edge-connected.

On the other hand, there are also locally finite graphs of high minimum degree and vertex-degree that have no finite highly edge-connected subgraphs. To see this, we reuse an example from the introduction: for given $r \in \mathbb{N}$, add some edges to each level $S_i$ of the $r$-regular tree $T^r$ so that in the obtained graph $\tilde{T}^r$ each $S_i$ induces a path. The only end of $\tilde{T}^r$ has infinite vertex- and edge-degree, and the vertices of $\tilde{T}^r$ have degree at least $r$. Now, for every finite subgraph $H$ of $\tilde{T}^r$ there is a last level of $\tilde{T}^r$ that contains a vertex $v$ of $H$. Then $v$ has degree at most 3 in $H$, and hence, $H$ is not 4-edge-connected.

Our second corollary of Theorem 3.1 describes how the graph $G$ decomposes into subgraphs that either are highly edge-connected or are so that all their subgraphs send many edges to the outside. For this, we have to push the lower bound on the degree of the vertices a little:

**Corollary 3.5.** *Let $k \in \mathbb{N}$, and let $G$ be a graph whose vertices have degree at least $4k+1$, and whose ends have edge-degree at least $2k+1$. Then there is a set $\mathcal{D}$ of disjoint $(k+1)$-edge-connected regions of $G$ such that $|E(H, G-H)| \geq \max\{4k, |H|\}$ for each non-empty subgraph $H$ of $G - \bigcup_{D \in \mathcal{D}} D$.*

For the proof, we need the following lemma:

**Lemma 3.6.** *Let $m \in \mathbb{N}$ and let $G$ be a graph such that each of its vertices has degree at least $m$. Then every non-empty region $H$ of $G$ with $|E(H, G-H)| < m$ contains at least $m+1$ vertices.*

*Proof.* We may assume that $m > 1$. Now, we can estimate the number of edges of $H$ in two ways. On one hand,

$$\|H\| \geq \frac{m|H| - |E(H, G-H)|}{2} > \frac{m}{2}(|H| - 1),$$

as by assumption, each vertex of $H$ has degree at least $m$ in $G$. On the other hand, $H$ cannot have more edges than the complete graph on $|H|$ vertices. This leaves us with the inequality $\frac{m}{2}(|H| - 1) < \frac{|H|(|H|-1)}{2}$, implying that $|H| > m$. $\square$

*Proof of Corollary 3.5.* Let $\mathcal{D}$ be an inclusion-maximal set $\mathcal{D}$ of disjoint $(k+1)$-edge-connected regions of $G$ (which again exists by Zorn's Lemma).

Observe that it suffices to show $|E(H, G-H)| \geq \max\{4k, |H|\}$ for induced connected non-empty subgraphs $H$ of $G - \bigcup_{D \in \mathcal{D}} D$, and consider such an $H$. If $H$ is infinite, then Theorem 3.3 and the (maximal) choice of $\mathcal{D}$ imply that $H$ is not a region of $G$, i.e. that $|E(H, G-H)|$ is infinite, as desired.

So assume that $H$ is finite. In the case that $|H| < 4k$, Lemma 3.6 ensures that $|E(H, G-H)| \geq 4k$. In the case that $|H| \geq 4k$, suppose that $|E(H, G-H)| < |H|$. Then, $H$ has average degree $d(H) \geq \delta_V - 1 \geq 4k$, and hence $H$ has a $k$-edge-connected subgraph by Theorem 1.1, contradicting the choice of $\mathcal{D}$. $\square$

# 4 High edge-degree does not force highly connected subgraphs or minors

For given $r \in \mathbb{N}$ we will construct a locally finite graph $G_r$ of minimum degree $r$ at the vertices and minimum edge-degree at least $r$ at the ends that has no 4-connected subgraph and no 6-connected minor. The idea is to 'thicken' the ends of the tree $T_r$, in the sense of augmenting their edge-degree, which we do by adding many edges but only a few vertices in order to keep the separators small.

We start with an infinite rooted tree $T_r$ in which each vertex sends $r$ edges to the next level. The graph $G_r$ will be obtained from $T_r$ in the following manner. Let $S_0$ consist of the root $r_0$ of $T_r$ and for $i \geq 1$ denote by $S_i$ the $i$-th level of $T_r$. Now, successively for $i \geq 1$, we shall add some vertices to $S_i$, which results in an enlarged $i$th level $S_i'$, and then add some edges between $S_i' - S_i$ and $S_{i+1}$.

For each vertex $x \in S_{i-1}$, add $r-1$ new vertices $v_1^x, v_2^x, \ldots, v_{r-1}^x$ to its neighbourhood $S^x = \{s_1^x, s_2^x, \ldots, s_r^x\}$ in $S_i$. Denote by $S_i'$ the set thus obtained from $S_i$. Then for each $j \leq r-1$ and each $x \in S_{i-1}$ add all edges between $v_j^x$ and $N_{S_{i+1}}(\{s_j^x, s_{j+1}^x\})$. This yields a graph $G_r$ on the disjoint union of sets $S_1', S_2', \ldots$ as depicted in Figure 1 for $r = 4$.

The ends of $G_r$ correspond to the ends of the underlying tree $T_r$, i.e. every two disjoint rays in $T_r$ belong to different ends of $G_r$, and each end of $G_r$ contains a ray from $T_r$. Indeed, two rays from $T_r$ which in $T_r$ are separated by $\bigcup_{i=0}^{j} S_i$ for some $j \in \mathbb{N}$, can be separated in $G_r$ by the set $\bigcup_{i=0}^{j+1} S_i$. On the other hand, every ray $R \subseteq G_r$ has, for any fixed $j \in \mathbb{N}$, a tail in exactly one of the components of $G - \bigcup_{i=0}^{j} S_i$. This tail meets $S^{x_j}$, for some $x_j \in S_j$. Hence $R$ is equivalent to the ray $x_0 x_1 x_2 \ldots \subseteq T_r$.

**Lemma 4.1.** *$G_r$ has minimum degree $r$ at the vertices and minimum edge-degree at least $r$ at the ends.*

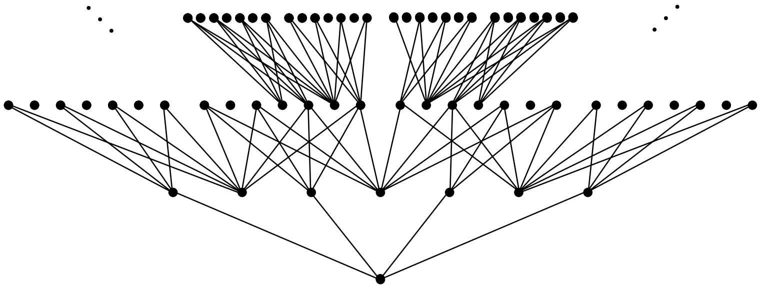

Figure 1: The graph $G_4$.

*Proof.* The definition of $G$ clearly ensures the desired degree at the vertices. We show that the ends of $G_r$ have edge-degree at least $r$ by constructing a set of $r$ edge-disjoint rays in each. Given an end $\omega$ of $G_r$, there is exactly one ray $R = r_0r_1r_2 \ldots \subseteq T_r$ in it (since the ends of $G_r$ correspond to those of $T_r$, as remarked above).

Now, construct $r-1$ edge-disjoint $\omega$-rays $R_i$, where $i = 1, \ldots, r-1$; these will also be edge-disjoint from $R$. Each $R_i$ starts in $r_0$, its second vertex is the $i$th neighbour of $r_0$ in $S_1$ which is unused by $R$. Next, it goes along some path that switches between $S_2$ and $S'_1$ until it reaches $r_2$. Note that we can choose these paths edge-disjoint for different $i$, for example by letting $R_i$ use only vertices $v \in S_2$ with $v = s_i^x$ for some $x \in S_1$ (or $v = r_2$). Similarly, we continue the $R_i$ going from $r_2$ to the $i$th unused neighbour in $S_3$, and from there along edge-disjoint paths to $r_4$, and so on. Since the $R_i$ agree on $r_0, r_2, r_4 \ldots$, they all belong to $\omega$. $\square$

Observe that every finite set $A$ of vertices can be separated from any end $\omega$ by at most three vertices (namely by the neighbours of the unique component of $G_r - S'_i$ that contains a ray in $\omega$, where $j$ is large enough so that $A \subseteq \bigcup_{i=0}^{j} S'_i$). Hence, each end of $G_r$ has vertex-degree at most 3.

In fact, Theorem 5.1 ensures that every graph of high minimum degree (at the vertices) has either an end of small vertex-degree or a highly connected subgraph. We shall see now that the latter is not the case for $G_r$.

**Lemma 4.2.** *$G_r$ has no 4-connected subgraph.*

*Proof.* Suppose otherwise, and let $H$ be a 4-connected subgraph of $G$. Let $i \in \mathbb{N}$ so that $V(H) \cap S'_i \neq \emptyset$. Now, if there is a vertex $v \in V(H) - \bigcup_{j=0}^{i+1} S'_i$, then it can be separated in $G_r$ (and thus also in $H$) from $V(H) \cap S'_i$ by at most three vertices, namely by the neighbours of the component of $G_r - S'_{i+1}$ that contains $v$. So, as $H$ is 4-connected, $V(H) - \bigcup_{j=0}^{i+1} S'_i$ must be empty. Then, there is a maximal $j \in \mathbb{N}$ such that $V(H) \cap S'_j \neq \emptyset$. But then by construction of $G_r$, any vertex in $V(H) \cap S'_j$ has degree at most three in $H$, contradicting the 4-connectedness of $H$. $\square$

It is only slightly more difficult to prove that $G_r$ has no highly connected minor:

**Lemma 4.3.** *$G_r$ has no 6-connected minor.*

*Proof.* Suppose that $G_r$ has a 6-connected minor $M$. Then there is an $n \in \mathbb{N}$ so that each branch-set of $M$ has a vertex in $\bigcup_{i=0}^{n} S'_i$. Since $M$ is 6-connected, each separator $T \subseteq \bigcup_{i=0}^{n} S'_i$ of $G_r$ with $|T| \leq 5$ leaves a component $C$ of $G_r - T$ such that $V(C) \cup T$ meets one and hence every branch-set of $M$. So as each $S'_i$ can be separated in $G_r$ from any component of $G - S'_i$ by at most three vertices, there is an $i < n$ such that each branch-set of $M$ meets $S'_i \cup S'_{i+1}$. Moreover, there is a vertex $x \in S_i$ such that for $S := N_{S_{i+1}}(x)$ we have that each branch-set of $M$ has a vertex in $S' := S \cup N_{S'_i}(S) \cup \{v_1^x, v_2^x, \ldots, v_{r-1}^x\}$. Then $|S' \cap S'_i| \leq 3$.

We claim that $M$ is also a minor of the finite graph $G'_r$ (see Figure 2) which

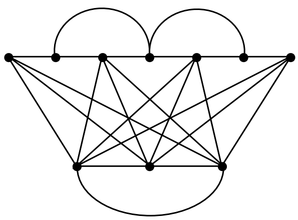

Figure 2: The graph $G'_4$ for $|S' \cap S'_i| = 3$.

is obtained from $G_r[S']$ by adding an edge between every two vertices that are neighbours of the same component of $G_r - S'$. Indeed, each component $C$ of $G_r - S'$ has at most three neighbours in $S'$. Hence, since $M$ is 6-connected, $C$ meets only (if at all) those branch-sets of $M$ that also meet $N_{S'}(C)$. It is easy to see that $M$ is still a minor of the graph we obtain from $G_r$ by deleting $C$ and adding all edges between vertices in $N_{S'}(C)$. Arguing analogously for the other components of $G_r - S'$, we see that $M$ is also a minor of $G'_r$.

As $|S' \cap S'_i| \leq 3$, all but at most 3 branch-sets of $M$ in $G'_r$ have all their vertices in $|S' \cap S'_{i+1}|$. Then these give rise to a 3-connected minor of $G'_r - S'_i$. But each non-trivial block of $G'_r - S'_i$ is a triangle and hence has no 3-connected minor, yielding the desired contradiction. □

Note that the two latter results are best possible, since $G_r$ has a 3-connected subgraph, the complete graph on 4 vertices, and a 5-connected minor, the complete graph on 6 vertices.

# 5 Forcing highly connected subgraphs

We shall finally prove our main result, which we restate:

**Theorem 5.1.** *Let $k \in \mathbb{N}$ and let $G$ be a graph so that each vertex has degree at least $\delta_V = 2k(k+3)$, and so that each end has vertex-degree at least $\delta_\Omega = 2k(k+1)+1$. Then every infinite region of $G$ has a $(k+1)$-connected region.*

For proving Theorem 5.1 we shall proceed at first similarly[4] as in the proof of Theorem 3.1 (resp. Theorem 3.3), until we arrive at an infinite region $C' \subseteq C$

[4]The situation here is a little more complicated, because a vertex-version of Lemma 3.2, replacing 'edges' with 'vertices' and 'cuts' with 'separators', fails, unless we make use of the high vertex-degree assumed in Theorem 5.1. To see this, take a ray $v_0v_1v_2\ldots$, to which we add all edges $v_0v_i$, and consider the region $D$ which consists of all $v_i$ but $v_1$.

with the property that $|\partial H| \geq \delta_\Omega$ holds for all regions $H \subsetneq C'$. This is achieved in Lemma 5.2 below.

But then, we see ourselves confronted with new difficulties. The region $C'$ need not be highly connected, or even 2-connected; the reason is that we lost control on the degrees of the vertices in $C'$. (And the situation only changes for the worse if instead of $C'$ we consider $C' - \partial C'$, which needs be neither connected, nor a region.)

Hence, we shall prefer a region $H \subseteq C'$ over $C'$, if the vertices of $\partial H$ have 'much' higher degree in $H$ than those of $\partial C'$ have in $C'$, even if $\partial H$ has 'slightly' greater cardinality than $\partial C'$. This will be formalised below.

Our combination of measurements on the suitability of $H$, on one hand $|\partial H|$, and on the other, $d_H(\partial H)$, is responsible for the quadratic lower bounds on the degrees which this proof of Theorem 5.1 yields. We shall see in Section 6 that these bounds are indeed close to best possible.

**Lemma 5.2.** *Let $G$ be a graph such that all its ends have vertex-degree at least $\delta_\Omega \in \mathbb{N}$. Let $C$ be an infinite region of $G$. Then there exists a profound region $C' \subseteq C$ for which one of the following holds:*

*(a) $C'$ is finite and $|\partial C'| < \delta_\Omega$, or*

*(b) $C'$ is infinite and $|\partial C''| \geq \delta_\Omega$ for every profound region $C'' \subsetneq C'$.*

*Proof.* Suppose otherwise. Then in particular, (b) does not hold for $C$, i.e. there is a profound region $C_1 \subsetneq C$ with $|\partial C_1| < \delta_\Omega$. We shall construct a sequence $C_1 \supsetneq C_2 \supsetneq \ldots$ of profound regions of $C$ so that $|\partial C_n| < \delta_\Omega$ for all $n \geq 1$.

So assume the region $C_n$ with small boundary to be found. We may assume $C_n$ to be such that $C_n - \partial C_n$ is connected and also, that $N(C_n - \partial C_n) = \partial C_n$. Indeed, if $C_n$ is not so, we can take any component $H$ of $C_n - \partial C_n$ (such a component exists, as $C_n$ is profound) and add $N(H) \subseteq \partial C_n$. Then $H \subseteq C_n$ is a profound region with the desired small boundary which we may use instead of $C_n$.

As we suppose that (a) does not hold for $C_n$, we can assume that $C_n$ is infinite. Since $C_n$ is not as required in (b), there is a profound region $C_{n+1} \subsetneq C_n$ with $|\partial C_{n+1}| < \delta_\Omega$. In this manner, we obtain an infinite sequence $(C_n)_{n\in\mathbb{N}}$.

Denote by $V$ the (possibly empty) set of those vertices $v$ that from some $j \in \mathbb{N}$ on appear in all $\partial C_n$ with $n \geq j$. Since all $\partial C_n$ have size at most $\delta_\Omega$, the set $V$ is finite. Furthermore, as $C_{n+1} \subsetneq C_n$ for every $n \geq 1$, and we chose the $C_n$ so that $C_n - \partial C_n$ is connected, we have that $\partial C_n = V$ for at most one $n \in \mathbb{N}$. Let $J$ be such that $V \subsetneq \partial C_n$ for all $n \geq J$.

Observe that for each $w \in \partial C_J - V$ there is an index $j$ such that $w \notin \partial C_n$ for all $n \geq j$. Hence, there is an index $J'$ so that $\partial C_J \cap \partial C_{J'} = V$. Set $C'_1 := C_J$, and set $C'_2 := C_{J'}$. Continuing in this manner, we arrive at an infinite subsequence $C'_1 \supsetneq C'_2 \supsetneq \ldots$ of profound regions of $C$, whose boundaries pairwisely meet only in $V$. Let us resume the properties which the regions $C'_n$ have, for each $n \in \mathbb{N}$:

(i) $|\partial C'_n| < \delta_\Omega$,

(ii) $C'_n - \partial C'_n$ is connected, and $N(C'_n - \partial C'_n) = \partial C'_n$,

(iii) $V \subsetneq \partial C'_n$, and

(iv) $\partial C'_{n+1} \subseteq (C'_n - \partial C'_n) \cup V$.

We claim that there is a ray $R$ that has a tail in each $C'_n$. Indeed, by (ii) and (iii), there exists for each $n \in \mathbb{N}$ a $(\partial C'_n - V)$–$(\partial C'_{n+1} - V)$ path $P_n$ such that each $P_{n+1}$ starts in the last vertex of $P_n$. By (iv), the paths $P_i$ are non-trivial, hence, their union is the desired ray $R$. Denote by $\omega$ the end of $G$ that contains $R$.

As, by assumption, $\omega$ has vertex-degree at least $\delta_\Omega$, there is a set $\mathcal{R}$ of $\delta_\Omega$ disjoint $\omega$-rays in $G$. The starting vertices of the rays in $\mathcal{R}$ lie either outside $C'_1 - \partial C'_1$, or have in $C'_1 - V$ a finite distance to $\partial C'_1 - V$. Hence, by (iv), there is an $N \in \mathbb{N}$ so that all rays of $\mathcal{R}$ start outside $C'_N - \partial C'_N$. But (being equivalent to $R$) each of these disjoint rays eventually enters $C'_N - \partial C'_N$, and thus meets $\partial C'_N$, a contradiction because by (i), $|\partial C'_N| < \delta_\Omega$. This completes the proof of Lemma 5.2. $\square$

We are now ready to prove our main result.

*Proof of Theorem 5.1.* Given an infinite region $C$ of $G$, we shall find a $(k+1)$-connected region $H \subseteq C$. Theorem 5.1 obviously holds for $k = 1$, since the ends of a tree have vertex-degree $1 < \delta_\Omega$. We can thus assume that $k > 1$.

Suppose there exists a profound finite region $D \subseteq C$ with $|\partial D| < \delta_\Omega$. Then $D - \partial D$ has minimum degree at least $\delta(D - \partial D) \geq \delta_V - \delta_\Omega + 1 = 4k$. Hence Theorem 1.1 yields a finite $(k+1)$-connected subgraph of $D \subseteq C$, and we are done. Let us therefore assume that there is no such region $D$.

We may thus apply Lemma 5.2 to obtain an infinite region $C' \subseteq C$ with the property that

$$|\partial C''| \geq \delta_\Omega \text{ for every profound region } C'' \subsetneq C'. \tag{1}$$

For a region $H \subseteq C'$ write

$$\Sigma_H := \sum_{v \in V(H)} \max\{0, \delta_V - d_H(v)\}.$$

Observe that this sum is finite, since all vertices of $H$ but the finitely many in $\partial H$ have degree at least $\delta_V$ in $H$. Now, choose an infinite region $H \subseteq C'$ such that $(k+3)|\partial H| + \Sigma_H$ is minimal.

Assume that there is a vertex $v \in V(H)$ that has degree at most $2k-3$ in $H$. Then clearly, $v \in \partial H$. Observe that $d_{H-v}(w) = d_H(w) - 1$ for each of the at most $2k-3$ neighbours $w$ of $v$ in $H$, and $d_{H-v}(w') = d_H(w')$ for all other vertices $w'$ in $H$. Therefore,

$$\begin{aligned}(k+3)|\partial(H-v)| + \Sigma_{H-v} &\leq (k+3)|\partial H| + (k+3)(2k-4) \\ &\quad + \Sigma_H + (2k-3) - (\delta_V - d_H(v)) \\ &< (k+3)|\partial H| + \Sigma_H + 2k(k+3) - \delta_V \\ &= (k+3)|\partial H| + \Sigma_H.\end{aligned}$$

So any infinite component of $H - v$ is a better choice than $H$, a contradiction. We thus have shown that

$$d_H(v) \geq 2(k-1) \text{ for each vertex } v \in V(H). \tag{2}$$

Let us prove now that $H$ is the desired $(k+1)$-connected region of $C$. Indeed, suppose otherwise. Then $H$ has a separator $T$ of cardinality at most $k$, which we may assume to be a minimal separator. Note that each such separator leaves a component $D$ of $H - T$ such that $H' := H - D$ is an infinite region of $C$.

Suppose that $|V(D) \cap \partial H| \geq \delta_\Omega - |T|$. Then we obtain for the infinite region $H' \subseteq C'$ that

$$\begin{aligned} |\partial H'| &= |(\partial H - V(D)) \cup T| \\ &\leq |\partial H| - |V(D) \cap \partial H| + |T| \\ &\leq |\partial H| - \delta_\Omega + 2k. \end{aligned}$$

Furthermore,

$$\begin{aligned} \Sigma_{H'} &\leq \Sigma_H + \sum_{v \in T} \max\{0, \delta_V - d_{H'}(v)\} \\ &\leq \Sigma_H + k\delta_V, \end{aligned}$$

and so

$$\begin{aligned} (k+3)|\partial H'| + \Sigma_{H'} &\leq (k+3)|\partial H| - (k+3)(\delta_\Omega - 2k) + \Sigma_H + k\delta_V \\ &< (k+3)|\partial H| - 2k^2(k+3) + \Sigma_H + 2k^2(k+3) \\ &= (k+3)|\partial H| + \Sigma_H, \end{aligned}$$

contradicting the choice of $H$.

Hence,

$$|V(D) \cap \partial H| < \delta_\Omega - |T|.$$

Thus for the region $\tilde{D} := G[V(D) \cup T] \subseteq C'$, we have

$$|\partial \tilde{D}| = |(V(D) \cap \partial H) \cup T| \leq |V(D) \cap \partial H| + |T| < \delta_\Omega.$$

Observe that $\tilde{D} \neq H$. So, by (1), the region $\tilde{D}$ is not profound, i.e. $V(\tilde{D}) = \partial \tilde{D}$, implying that $V(D) \subseteq \partial H$. In particular, $|D| < \delta_\Omega - |T|$. Now, for any vertex $v \in V(D)$, we can estimate its degree in $H$ as follows.

$$d_H(v) \leq |(D \cup T) - \{v\}| < \delta_\Omega - 1 = \delta_V - 4k.$$

Then $\delta_V - d_H(v) > 4k$, implying that

$$\begin{aligned} \Sigma_{H'} &\leq \Sigma_H - \sum_{v \in V(D)} \max\{0, \delta_V - d_H(v)\} + \sum_{v \in T} (d_H(v) - d_{H'}(v)) \\ &< \Sigma_H - 4k|D| + |T||D| \\ &\leq \Sigma_H - 3k. \end{aligned}$$

On the other hand, (2) ensures that $|D| \geq 1 + 2(k-1) - |T| \geq k - 1$. So

$$|\partial H'| \leq |\partial H| - |D| + |T| \leq |\partial H| + 1,$$

and thus (as $k > 1$ by assumption)

$$\begin{aligned} (k+3)|\partial H'| + \Sigma_{H'} &< (k+3)|\partial H| + (k+3) + \Sigma_H - 3k \\ &\leq (k+3)|\partial H| + \Sigma_H, \end{aligned}$$

again contradicting the choice of $H$. $\square$

We finish this section with two corollaries of Theorem 5.1. The proof of the first is analogous to that of Corollary 3.4.

**Corollary 5.3.** *Let $k \in \mathbb{N}$ and let $C$ be an infinite region of a graph $G$ of minimum degree at least $2k(k+3)$ at the vertices and minimum vertex-degree at least $2k(k+1)+1$ at the ends. Then $C$ has either infinitely many disjoint finite $(k+1)$-connected regions or an infinite $(k+1)$-connected region.*

Again, these two configurations need not both exist, as the examples following Corollary 3.4 illustrate. (Observe that if a graph has no $k$-edge-connected subgraph then it clearly has no $k$-connected subgraph.)

The second corollary of Theorem 5.1 is an analogue of Corollary 3.5.

**Corollary 5.4.** *Let $k \in \mathbb{N}$, and let $G$ be a graph whose vertices have degree at least $\delta_V = 2k(k+3)$ and whose ends have vertex-degree at least $\delta_\Omega = 2k(k+1)+1$. Then there is a set $\mathcal{D}$ of disjoint $(k+1)$-connected regions of $G$ such that $|\partial H| \geq \max\{\delta_\Omega, \frac{k-2}{k}|H|+1\}$ for each profound subgraph $H$ of $G - \bigcup_{D\in\mathcal{D}} D$.*

*Proof.* Similarly as in the proof of Corollary 3.5, take an inclusion-maximal set $\mathcal{D}$ of disjoint $(k+1)$-connected regions of $G$, and observe that we only need to consider induced connected profound subgraphs $H$ of $G - \bigcup_{D\in\mathcal{D}} D$. So let $H$ be a such. If $H$ is infinite, then Theorem 5.1 and the choice of $\mathcal{D}$ imply that $H$ is not a region, i.e. that $|\partial H|$ is infinite, as desired.

So assume that $H$ is finite. Then $|\partial H| \geq \delta_\Omega$, as otherwise $H - \partial H$ has minimum degree $\delta(H - \partial H) \geq \delta_V - \delta_\Omega + 1 \geq 4k$, and hence $H$ has a $(k+1)$-connected subgraph by Theorem 1.1, contradicting the choice of $\mathcal{D}$.

Also, $|\partial H| > \frac{k-2}{k}|H|$. Indeed, suppose otherwise. Then $H$ has average degree

$$d(H) \geq \frac{\delta_V|H - \partial H| + |\partial H|}{|H|} \geq \delta_V - (\delta_V - 1)\frac{|\partial H|}{|H|} \geq \frac{2\delta_V + k - 2}{k} \geq 4k.$$

Thus again, Theorem 1.1 yields a $(k+1)$-connected subgraph of $H$, a contradiction to the choice of $\mathcal{D}$. □

# 6 Linear degree bounds are not enough

Unlike in Mader's original theorem, and in Theorem 3.1, the bounds on the degrees and vertex-degrees we require in Theorem 5.1 are quadratic in $k$. It seems that our method of proof cannot yield better bounds, because the region $H$ we find has to be best possible in two ways: small boundary on one hand, high in-degree of its vertices on the other. But the quadratic bounds we give are in fact not far from best possible: a minimum degree and vertex-degree only linear in $k$ is insufficient to ensure $(k+1)$-connected subgraphs.

**Theorem 6.1.** *For each $k = 5\ell$, where $\ell \in \mathbb{N}$, there exists a locally finite graph whose vertices have degree at least $2^\ell$, whose ends have vertex-degree at least $\ell \log \ell$, and which has no $(k+1)$-connected subgraph.*

*Proof.* Set $m := \lceil \log \ell \rceil$. The vertex set of our graph $G$ will be that of a tree $T$, which is rooted in $v_0$. The root has $2\ell$ neighbours in the first level $S_1$ of $T$, and for $i \geq 1$ each vertex in the $i$th level $S_i$ sends two edges to the next level $S_{i+1}$.

Set $S_0 := \{v_0\}$. Observe that the tree $T$ induces an order $\leq$ on the vertex set $V(G) = V(T)$, that is, $x \leq y$ for $x, y \in V(G)$ if and only if $x$ lies on the unique $v_0$–$y$ path in $T$.

Now, for each $i \geq 0$ and each $x \in S_i$, add to $T$ all edges $xy$, where $y \in S_{i+\ell}$ and $y \geq x$. Note that each $x \neq v_0$ has exactly $2^\ell$ such 'new neighbours' $y$ (while $v_0$ has at least that many). Hence, in the thus obtained graph $G'$, each vertex $v$ has degree $d_{G'}(v) \geq 2^\ell$.

In order to achieve a high vertex-degree in the ends of the graph, we shall add a few more edges to $G'$. For this, let us have a closer look at $T$. For $j \in \mathbb{N}$, we inductively define sets $S^{(s)} \subseteq S_j$ for each 01-string of length $j \geq 1$. Divide $S_1$ arbitrarily into two sets $S^{(0)}, S^{(1)}$ of equal size $(= \ell)$. Then for each $j \geq 2$, and for each 01-string $s$ of length $j-1$, partition the neighbourhood of $S^{(s)}$ in $S_j$ into two sets $S^{(s0)}, S^{(s1)}$ of equal size $(= \ell)$, in a way that the neighbourhoods of $S^{(s0)}$ and $S^{(s1)}$ in $S_{j-1}$ are disjoint. Then $S_j$ is the disjoint union of all $S^{(s)}$, where $s$ varies over all 01-strings of length $j$. Now, for each 01-string $s$ of any length, and for each 01-string $t$ of length $m$ match $S^{(s0)}$ with $S^{(s1t)}$, and match $S^{(s1)}$ with $S^{(s0t)}$.

This yields a graph $G$, which we claim to be the one desired. Indeed, we have seen that already in $G'$ the vertices have the required degree. Let us now investigate the end structure of $G$.

We claim that $G$ does not have 'more' ends than $T$, i.e. every end of $G$ contains a ray from $T$. Indeed, consider a ray $R$ of $G$: we shall show that there is a ray in $T$ which is equivalent to $R$. Let $C_0$ be the (unique) component of $G - \bigcup_{i=1}^{m+1} S_i$ that contains a tail of $R$. There is a path $P_0$ in $T$ that connects this tail with the unique vertex $x_0 \in S_1$ for which holds $x_0 \leq c$ for all $c \in C_0$. Now, choose $j$ large enough so that $V(P_0) \subseteq \bigcup_{i=0}^{j-1} S_i$, and let $C_1$ be the component of $G - \bigcup_{i=1}^{j+m} S_i$ that contains a tail of $R$. Again, there is a path $P_1$ in $T$ that connects this tail with the unique vertex $x_1 \in S_j$ for which holds $x_1 \leq c$ for all $c \in C_1$.

Continuing in this manner, we obtain an infinite set of disjoint paths $P_i$, where each $P_i$ connects $x_i$ with $V(R)$. Clearly, since $x_0 \leq x_1 \leq x_2 \leq \ldots$, there is a ray $R'$ in $T$ that contains all vertices $x_i$. The ray $R'$ cannot be finitely separated from $R$, and thus is equivalent to $R$. Hence, every end of $G$ contains a ray of $T$, as desired.[5]

Next, let us show that the ends of $G$ have vertex-degree at least $\ell m$. Given an end $\omega \in \Omega(G)$, and a ray $R = v_0 v_1 v_2 \ldots \in \omega$, with $R \subseteq T$ and $v_i \in S_i$ for all $i$, we shall find a set of disjoint rays $R^i_j \in \omega$, where $i = 1, 2, \ldots, m$; and $j = 1, 2, \ldots, \ell$. These rays will exclusively use edges from $E(G) - E(G')$. Let $s = s_1 s_2 s_3 \ldots$ be a 01-string of infinite length so that $v_n \in S^{(s_1 s_2 \ldots s_n)}$ for each $n \geq 1$. Denote by $S(n)$ the set $S^{(s_1 s_2 \ldots s_{n-1}(1-s_j))}$. Now, for fixed $i \in \{1, \ldots, m\}$, the $\ell$ disjoint rays $R^i_j$ will pass through all sets $S(n)$, where $n = i, i+m, i+2m, i+3m, \ldots$, using the $S(n)$–$S(n+m)$ edges of the matching from the definition of $G$. This is illustrated in Figure 3.

We thus obtain the desired rays $R^i_j$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, \ell$. Observe that for each $i, j$, there are infinitely many disjoint $V(R)$–$V(R^i_j)$ paths in $T$, namely those that connect $v_n$ with the vertex of $R^i_j$ in $S(n+m)$. Hence the $R^i_j$ are equivalent to $R$, and hence, $\omega$ has vertex-degree at least $\ell m$.

[5]Moreover, as any two distinct rays of $T$ that start in $v_0$ can be finitely separated in $G$, we have that the ends of $G$ correspond to the ends of the tree $T$.

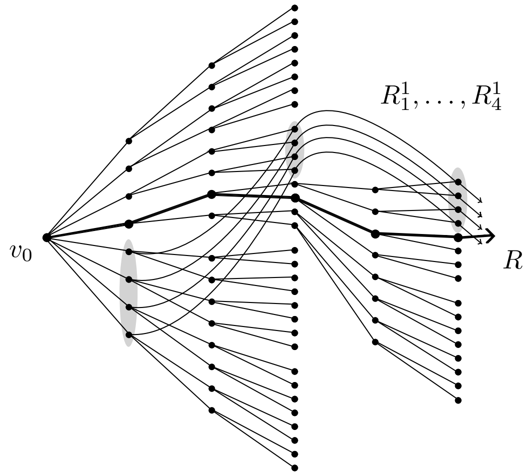


Figure 3: The equivalent rays $R_1^1, \ldots, R_4^1$ and $R$ in the underlying tree $T$, for $\ell = 4$.

Let us finally prove that the graph $G$ has no $(k+1)$-connected subgraph. Indeed, suppose otherwise, and let $H \subseteq G$ be $(k+1)$-connected. Now, for any given 01-string $s$, denote by $T^{(s)}$ the set of all vertices $y$ that are comparable in $T$ with one of the elements of $S^{(s)}$, and which, in the case that $y < x \in S^{(s)}$, in $T$ have distance less than $\ell$ to $x$. Formally,

$$\begin{aligned} T^{(s)} :=& \{y \in V(G) : \textit{ there is an } x \in S^{(s)} \textit{ such that } y < x \textit{ and } d_T(x,y) < \ell\} \\ & \cup \{y \in V(G) : \textit{ there is an } x \in S^{(s)} \textit{ such that } y \geq x\}. \end{aligned}$$

We claim that for each $n \geq 1$

$$\textit{there is a 01-string } s \textit{ of length } n \textit{ so that } V(H) \subseteq T^{(s)}. \tag{3}$$

Then, for every $i \in \mathbb{N}$ and for any vertex $v \in S_i$, we may apply (3) with $n = i + \ell$ to obtain that $v \notin V(H)$. Hence $H = \emptyset$, a contradiction.

It remains to show (3), which we do using induction on $n$. For $n = 1$, observe that $S_0 \cup S_1$ separates in $G$ the sets $T^{(0)} - (S_0 \cup S_1)$ and $T^{(1)} - (S_0 \cup S_1)$. Hence, because $|S_0 \cup S_1| \leq k$, for either $s = 0$ or $s = 1$ we have that $V(H) \subseteq T^{(s)} \cup (S_0 \cup S_1) = T^{(s)} \cup S_1$. Furthermore, as by construction of $G$ the vertices in $S^{(1-s)}$ each send only $2^m \leq 2\ell \leq k$ edges to $T^{(s)}$, it follows that $V(H) \subseteq T^{(s)}$, as desired.

For $n > 1$, we proceed similarly. The induction hypothesis provides us with a string $s'$ of length $n-1$ such that $V(H) \subseteq T^{(s')}$. Now, the set

$$S := T^{(s')} \cap \bigcup_{i=n-\ell+1}^{n} S_i$$

has size at most

$$(\ell - m) + \sum_{i=0}^{m} 2^i + |T^{(s')} \cap S_n| \leq \ell + 2\ell + 2\ell = k.$$

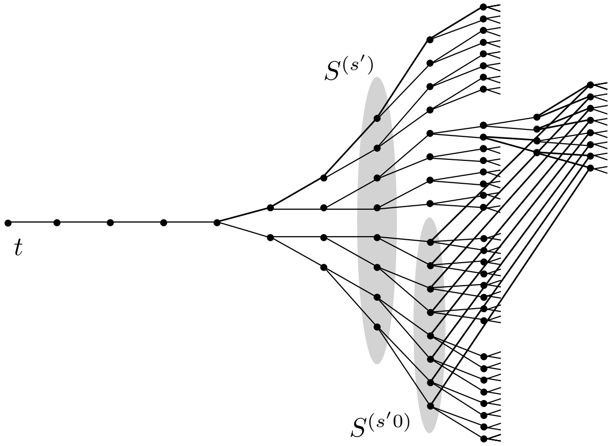


Figure 4: The subgraph of $G$ induced by $T^{(s')}$, for $\ell = 8$. For simplicity, the edges of $G' - E(T)$ are not drawn.

Moreover, $S$ separates $G[T^{(s')}]$ into the three sets $T^{(s')} \cap S_{n-\ell}$, $T^{(s'0)} - S$ and $T^{(s'1)} - S$, the first of which consists of one vertex $t$ only. Thus, for either $s = s'0$ or $s = s'1$, say for $s = s'0$, we have that $V(H) \subseteq T^{(s)} \cup S$.

Observe that we can write

$$S - T^{(s)} = S^{(s'1)} \cup U$$

where

$$U = (T^{(s')} - T^{(s)}) \cap \bigcup_{i=n-m}^{n-1} S_i.$$

Now, by construction of $G$, each vertex of $U$ has at most 3 neighbours in $T^{(s)} \cup S$. Therefore, $V(H) \cap U = \emptyset$. Moreover, as the vertices in $S^{(s'1)}$ each send only $2^m \leq k$ edges to $T^{(s)} \cup S^{(s'1)}$, it follows that $V(H) \cap S^{(s'1)} = \emptyset$, and hence, $V(H) \subseteq T^{(s)}$, as desired. This completes the proof of (3), and thus the proof of the theorem. $\square$

# References


[1] H. Bruhn and M. Stein. On end degrees and infinite circuits in locally finite graphs. To appear in Combinatorica.

[2] R. Diestel. The cycle space of an infinite graph. *Comb., Probab. Comput.*, 14:59–79, 2005.

[3] R. Diestel. *Graph Theory* (3rd ed.). Springer-Verlag, 2005.

[4] R. Halin. Über die Maximalzahl fremder unendlicher Wege in Graphen. *Math. Nachr.*, 30:63–85, 1965.

[5] W. Mader. Existenz $n$-fach zusammenhängender Teilgraphen in Graphen mit genügend hoher Kantendichte. *Abh. Math. Sem. Univ. Hamburg*, 37:86–97, 1972.